Math 415
Lecture 2

Sec 1.3

First Chapter is all about solving linear equations
Main technique is one of reducing a system to a simpler diagonal system and then using back substitution

$$\begin{aligned} x + 2y + z &= 2 \\ 2x + 6y + z &= 7 \\ x + y + 4z &= 3 \end{aligned} \quad \rightarrow \quad \begin{aligned} x + 2y + z &= 2 \\ 2y - z &= 3 \\ \tfrac{5}{2} z &= \tfrac{5}{2} \end{aligned} \quad \begin{aligned} &\longrightarrow x = -3 \\ &\longrightarrow y = 2 \\ &\longrightarrow z = 1 \end{aligned}$$

Matrix notation is introduced to help us shorten the writing

Gaussian Elimination

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 1 & 1 & 4 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & -1 & 3 & 1 \end{array}\right)$$

pivot

pivot

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \end{array}\right)$$

upper triangular

All done by "elementary row operations"

Note:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\Big\downarrow \text{ add } -2\times \text{1st row to } 2^{nd} \text{ row}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{elementary matrix } E \text{ of the first type}} \underbrace{\left(\begin{array}{ccc|c} 1 & 2 & 1 & \cancel{2} \\ 2 & 6 & 1 & \cancel{7} \\ 1 & 1 & 4 & \cancel{3} \end{array}\right)}_{A} = \left(\begin{array}{ccc|c} 1 & 2 & 1 & \cancel{2} \\ 0 & 2 & -1 & \cancel{3} \\ 1 & 1 & 4 & \cancel{3} \end{array}\right)$$

call this E_1

$E_1 A$ is

Next operation: add $-1 \times$ 1st row to 3^{rd} row:

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad E_2 E_1 A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Set $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$

$$\Rightarrow \quad E_3 E_2 E_1 A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{5}{2} \end{pmatrix} = U \quad \text{upper triangular matrix}$$

Set $L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$

Why is it the case that $L_1 E_1 = I$, $E_1 L_1 = I$, $L_2 E_2 = I$, $E_2 L_2 = I$, etc?

Set $L = L_1 L_2 L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \end{pmatrix}$ "special lower triangular".

This uses the observation:
Product of lower triangular matrices is lower triangular

$$\underbrace{L_3 E_3}_{I} E_2 E_1 A = L_3 U$$

$$\underbrace{L_2 E_2}_{I} E_1 A = L_2 L_3 U$$

$$\underbrace{L_1 E_1}_{I} A = L_1 L_2 L_3 U$$

$$A = LU \qquad \text{(LU factoring of a matrix)}$$

A square matrix is "regular" if it has an LU decomposition

$$Ax = b \Leftrightarrow LUx = b \Leftrightarrow \begin{array}{l} Ux = c \leftarrow \text{solve for } x \\ \text{iff } Lc = b \leftarrow \text{solve for } c \end{array}$$

forward + back subst'n problems.

$Ax = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Example $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & -3 & -1 \end{pmatrix} \to \underbrace{\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{U}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$Lc = b \quad \text{ie} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{aligned} c_1 &= 1 \longrightarrow c_1 = 1 \\ 2c_1 + c_2 &= 2 \longrightarrow c_2 = 0 \\ c_1 - c_2 + c_3 &= 2 \longrightarrow c_3 = 1 \end{aligned}$$

Forward Substitution

$$Ux = c \quad \text{ie} \quad \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \longrightarrow x_1 = 1 \\ 3x_2 &= 0 \longrightarrow x_2 = 0 \\ -x_3 &= 1 \longrightarrow x_3 = -1 \end{aligned}$$

$$\text{ie} \quad x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$