Math 415 Lecture 21

Chapter 5 Orthogonality

Sec 5.1

mutually orthogonal

Def'n A basis $u_1, \ldots, u_n$ of V is orthogonal if $\langle u_i, u_j \rangle = 0$ for all $i \neq j$. The basis is orthonormal if it is orthogonal and $\|u_i\| = 1$ for all i.

Example The standard basis $e_1, \ldots, e_n$ is orthonormal

Example

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Using the dot product, these are orthogonal

Set $u_1 = \frac{1}{\|v_1\|} v_1$, $u_2 = \frac{1}{\|v_2\|} v_2$, $u_3 = \frac{1}{\|v_3\|} v_3$ "unit vectors"

These now have length 1, ie $\|u_1\| = \sqrt{\frac{v_1}{\|v_1\|} \cdot \frac{v_1}{\|v_1\|}} = \frac{1}{\|v_1\|} \sqrt{v \cdot v} = \frac{\|v_1\|}{\|v_1\|} = 1$.

Note

$$\|v_1\| = \sqrt{1+4+1} = \sqrt{6}, \quad \|v_2\| = \sqrt{1+4} = \sqrt{5}, \quad \|v_3\| = \sqrt{25+4+1} = \sqrt{30}$$

So $u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, etc.

Are they linearly indep?

Thm Any mutually orthogonal set of non-zero vectors is linearly indep.

Pf Let $v_1, \ldots, v_n$ be mutually orthogonal and consider

$$c_1 v_1 + \cdots + c_n v_n = 0.$$

Take the inner product with v_i:

all are zero except i^{th}

$$0 = \langle v_i, 0 \rangle = \langle v_i, c_1 v_1 + \cdots + c_n v_n \rangle = c_1 \langle v_i, v_1 \rangle + \cdots + c_n \langle v_i, v_n \rangle$$

$$= c_i \langle v_i, v_i \rangle = c_i \|v_i\|^2$$

$$\Rightarrow c_i = 0 \text{ since } v_i \neq 0.$$

__Example__ In $P^{(2)}$ with $\langle p,q\rangle = \int_0^1 p(x)q(x)\,dx$

consider

$$p_1(x) = 1,\quad p_2(x) = x-\tfrac{1}{2},\quad p_3(x) = x^2 - x + \tfrac{1}{6}$$

Then $\langle p_1, p_2\rangle = \int_0^1 1\cdot(x-\tfrac{1}{2})\,dx = \frac{x^2}{2} - \frac{1}{2}x \Big|_0^1 = \frac{1}{2} - \frac{1}{2} = 0$

$\langle p_1, p_3\rangle = \int_0^1 1\cdot(x^2-x+\tfrac{1}{6})\,dx = \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{6}x\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0$

$\langle p_2, p_3\rangle = \cdots = 0$

Hence these are linearly indep and hence an orthogonal basis of $P^{(2)}$. Let's make it orthonormal.

$$\|p_1\| = \sqrt{\int_0^1 1\cdot 1\,dx} = 1$$

$$\|p_2\| = \sqrt{\int_0^1 (x-\tfrac{1}{2})(x-\tfrac{1}{2})\,dx} = \sqrt{\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{4}x\Big|_0^1} = \sqrt{\frac{1}{3}+\frac{1}{4}-\frac{1}{2}} = \frac{1}{\sqrt{12}}$$

$$\|p_3\| = \cdots = \frac{1}{\sqrt{180}}$$

So set

$$u_1(x) = 1,\quad u_2(x) = \sqrt{12}\,(x-\tfrac{1}{2}),\quad u_3(x) = \sqrt{180}\,(x^2 - x + \tfrac{1}{6})$$

How do we compute the "coordinates" of a vector v relative to an orthonormal basis $u_1, \dots, u_n$:

$$v = c_1 u_1 + \cdots + c_n u_n \qquad \text{(the } c_i\text{'s are the coordinates)}$$

$$\Rightarrow \quad \langle u_i, v\rangle = c_1\langle u_i, u_1\rangle + \cdots + c_n\langle u_i, u_n\rangle$$

all zero except i^{th}

$$= c_i \langle u_i, u_i\rangle$$

$$= c_i \|u_i\|^2 = c_i$$

Also $\|v\| = \langle \sum_i c_i u_i, \sum_j c_j u_j \rangle^{1/2}$

$$= \Big(\sum_{i,j} c_i c_j \langle u_i, u_j \rangle \Big)^{1/2}$$

$$= \Big(\sum_i c_i^2 \|u_i\|^2 \Big)^{1/2}$$

$$= \sqrt{c_1^2 + \cdots + c_n^2}.$$

Example: See handout! How do we make these unit vectors?

Example: What are the coordinates of $p(x) = x^2+x+1$ relative to basis u_1, u_2, u_3 from $P^{(2)}$ earlier?

$$x^2+x+1 = p(x) = c_1 u_1(x) + c_2 u_2(x) + c_3 u_3(x)$$

where

$$c_1 = \langle x^2+x+1, u_1(x) \rangle$$

$$= \int_0^1 (x^2+x+1) \cdot 1 \, dx = \cdots = \frac{11}{6}$$

$$c_2 = \langle x^2+x+1, u_2(x) \rangle$$

$$= \int_0^1 (x^2+x+1) \sqrt{12} \left(x - \tfrac{1}{2}\right) dx = \cdots = \frac{1}{\sqrt{3}}$$

$$c_3 = \langle x^2+x+1, \sqrt{180}\left(x^2 - x + \tfrac{1}{6}\right) \rangle$$

$$= \cdots = \frac{1}{\sqrt{180}}$$

Thus $x^2+x+1 = \frac{11}{6} + 2\left(x - \frac{1}{2}\right) + \left(x^2 - x + \frac{1}{6}\right)$

Fourier Series

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$$

$$\langle p(x), q(x)\rangle = \int_{-\pi}^{\pi} p(x) q(x)\, dx.$$

You can show that

$$\langle \cos kx, \cos lx\rangle = \int_{-\pi}^{\pi} \cos kx \cos lx\, dx = \begin{cases} 0 & \text{if } k \neq l \\ 2\pi & \text{if } k = l = 0 \\ \pi & \text{if } k = l \neq 0 \end{cases}$$

$$\langle \sin kx, \sin lx\rangle = \int_{-\pi}^{\pi} \sin kx \sin lx\, dx = \begin{cases} 0 & k \neq l \\ \pi & k = l \neq 0 \end{cases}$$

$$\langle \sin kx, \cos lx\rangle = \int_{-\pi}^{\pi} \cos lx \sin kx\, dx = 0 \quad \forall k, l.$$

Then if we write

$$p(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$, we have

$$a_0 = \frac{\langle p, 1\rangle}{\|1\|^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)\, dx$$

$$a_k = \frac{\langle p, \cos kx\rangle}{\|\cos kx\|^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} p(x) \cos kx\, dx$$

$$b_k = \frac{\langle p, \sin kx\rangle}{\|\sin kx\|^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} p(x) \sin kx\, dx.$$

Fourier Series!