Math 415 Lecture 22

Section 5.2 The Gram-Schmidt Process

A way to turn a basis in to an orthonormal basis. Begin with a basis

$\{w_1, \dots, w_n\}$

Set $\boxed{v_1 = w_1}$

$v_2 = w_2 - cv_1$

and choose c so that

$$\langle v_1, v_2\rangle = 0 = \langle v_1, w_2\rangle - c\langle v_1, v_1\rangle$$

$$\Rightarrow \quad c = \frac{\langle v_1, w_2\rangle}{\langle v_1, v_1\rangle}$$

ie

$$\boxed{v_2 = w_2 - \frac{\langle v_1, w_2\rangle}{\langle v_1, v_1\rangle} v_1}$$

Now set

$v_3 = w_3 - c_1 v_1 - c_2 v_2$

and coose c_1, c_2 so that v_3 is $\perp$ to both v_1 and v_2:

$$\langle v_3, v_1\rangle = 0 = \langle w_3, v_1\rangle - c_1\langle v_1, v_1\rangle + 0$$

$$\langle v_3, v_2\rangle = 0 = \langle w_3, v_2\rangle - 0 - c_2\langle v_2, v_2\rangle$$

ie $c_1 = \dfrac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle}$, $c_2 = \dfrac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle}$

So

$$\boxed{v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2}$$

Keep going!

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j\rangle}{\|v_j\|^2} v_j$$

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

so $\langle v_1, w_2\rangle = 1 - 2 = -1$

$\langle v_1, v_1\rangle = 1 + 1 + 1 = 3$

$$\text{so} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{-1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 1/3 \\ 5/3 \end{pmatrix}$$

$\langle w_3, v_1\rangle = 2 - 2 - 3 = -3$

$\langle w_3, v_2\rangle = \frac{8}{3} - \frac{2}{3} + 5 = 7$

$\langle v_2, v_2\rangle = \frac{16 + 1 + 25}{9} = \frac{42}{9} = \frac{14}{3}$

So

$$v_3 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} - \frac{-3}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{7}{\frac{14}{3}}\begin{pmatrix} 4/3 \\ 1/3 \\ 5/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -3/2 \\ -1/2 \end{pmatrix}$$

Now normalize: $\|v_1\| = \sqrt{3}$, $\|v_2\| = \sqrt{\frac{14}{3}}$

$\|v_3\| = \sqrt{\frac{7}{2}}$, so

$$u_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \\ \frac{5}{\sqrt{42}} \end{pmatrix}, \quad u_3 = \begin{pmatrix} 2/\sqrt{14} \\ -3/\sqrt{14} \\ -1/\sqrt{14} \end{pmatrix}$$

is our orthonormal set.

EXAMPLE 5.14

Here is a typical problem: find an orthonormal basis, with respect to the dot product, for the subspace $V \subset \mathbb{R}^4$ consisting of all vectors which are orthogonal to the given vector $\mathbf{a} = (1, 2, -1, -3)^T$. The first task is to find a basis for the subspace. Now, a vector $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ is orthogonal to $\mathbf{a}$ if and only if

$$\mathbf{x} \cdot \mathbf{a} = x_1 + 2x_2 - x_3 - 3x_4 = 0.$$

Solving this homogeneous linear system by the usual method, we find that the variables are x_2, x_3, x_4, and so a (non-orthogonal) basis for the subspace is

$$\mathbf{w}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{w}_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

To obtain an orthogonal basis, we apply the Gram–Schmidt process. First,

$$\mathbf{v}_1 = \mathbf{w}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The next element is

$$\mathbf{v}_2 = \mathbf{w}_2 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-2}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 1 \\ 0 \end{pmatrix}.$$

The last element of our orthogonal basis is

$$\mathbf{v}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\, \mathbf{v}_1 - \frac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\, \mathbf{v}_2$$

$$= \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{-6}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\frac{3}{5}}{\frac{6}{5}} \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}.$$

An orthonormal basis can then be obtained by dividing each $\mathbf{v}_i$ by its length:

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{pmatrix}.$$