Math 415 Lecture 23

Chapter 5.3 Orthogonal Matrices

<u>Def'n</u> A square matrix Q is called an <u>orthogonal matrix</u> if

$$Q^T Q = I$$

ie the inverse of Q is its transpose.

<u>Prop'n</u> A matrix is orthogonal iff its columns form an orthonormal basis relative to the Euclidean (dot) product.

<u>Pf</u> $Q = (u_1 \dots u_n) \Rightarrow$

$$Q^T Q = \begin{pmatrix} u_1^T \\ \vdots \\ u_n^T \end{pmatrix} (u_1 \cdots u_n) = \left(u_i^T u_j \right)$$

So $u_i^T u_j = 0$ if $i \neq j$, $u_i^T u_i = \|u_i\|^2 = 1$. <u>Done</u>

<u>Example</u> Find all 2×2 orthog matrices

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow a^2+c^2=1,\ b^2+d^2=1,\ ab+cd=0$$

Take $a = \cos\theta$, $b = \cos\psi$

$c = \sin\theta$, $d = \sin\psi$

$$\cos\theta\cos\psi + \sin\theta\sin\psi = 0$$

$$\cos(\theta-\psi) = 0$$

$$\Rightarrow \theta - \psi = \pm\frac{\pi}{2}$$

Thus either $b = -\sin\theta, d = \cos\theta$ or $b = \sin\theta, d = -\cos\theta$

$$\Rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad 0 \le \theta < 2\pi.$$

Note: If we think of $u = Qx$ as a transformation taking one vector x and producing another vector u, then we can think of Q as a trans/n.
Let

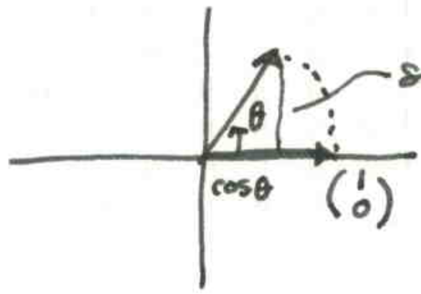
$$u = Qx, \quad v = Qy.$$

Then
$$\|u\|^2 = \|Qx\|^2 = Qx \cdot Qx = (Qx)^T Qx$$
$$= x^T \underbrace{Q^T Q}_{I} x$$
$$= x \cdot x = \|x\|^2 \text{ ie } \|u\| = \|x\|$$

Similarly $\|v\| = \|y\|$.
So Q does not change lengths. Also
$$u \cdot v = Qx \cdot Qy = (Qx)^T Qy = x^T Q^T Q y = x^T y = x \cdot y$$
So
$$\frac{u \cdot v}{\|u\| \cdot \|v\|} = \frac{x \cdot y}{\|x\| \, \|y\|}$$

ie angle between u and v = angle between x and y.

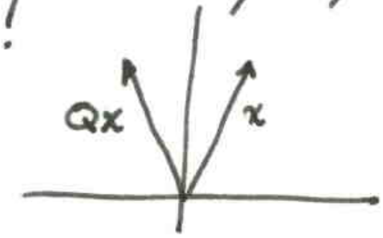
$$\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{Q} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



<u>Properties of Orthog Matrices</u>

<u>Prop 1</u> If Q is orthog, then $\det Q = \pm 1$. Why?
We say Q is "proper" orthogonal if $\det Q = 1$.
Proper Q's represent pure rot'ns & improper ones generally involve a reflection!

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{Q} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$$



$\Rightarrow \det Q = -1.$

<u>Prop 2</u>: If Q_1, Q_2 are orthog, so is $Q_1 Q_2$.

Talk about the QR factoriz'n if time:

$$w_1 = r_{11} u_1 \,, \qquad r_{11} = \|w_1\| \,, \text{etc}$$

$$w_2 = r_{21} u_1 + r_{22} u_2$$

$$w_3 = r_{31} u_1 + r_{32} u_2 + r_{33} u_3 \,, \text{etc}$$

$$\Rightarrow A = (w_1\ w_2 \cdots w_n) = (r_{11} u_1 \,,\ r_{21} u_1 + r_{22} u_2 \,, \ldots\ldots)$$

$$= (u_1\ u_2 \cdots u_n) \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & & r_{2n} \\ \vdots & 0 & & \vdots \\ 0 & 0 & & r_{nn} \end{pmatrix}$$

$$= \underset{\uparrow \atop \text{orthog}}{Q}\ \underset{\uparrow \atop \text{upper triangular}}{R}$$