Math 415 Lecture 24

Sec 5.5 (Fri March 14)

Let $W \subset V$ be a finite dimensional subspace of a vector space V.

Def'n A vector $z \in V$ is orthogonal to W if $\langle z, w \rangle = 0$ for every vector w in W.

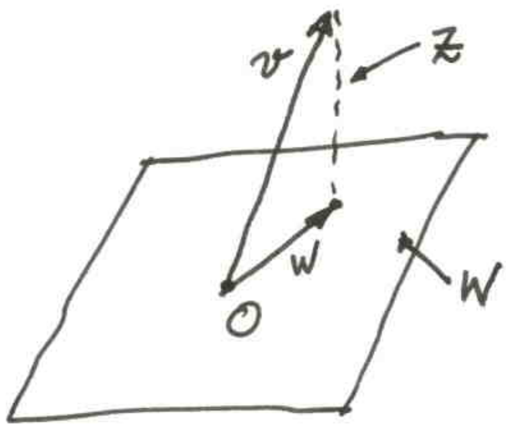
Def'n The orthogonal projection of $v \in V$ onto W is the element w in W for which $z = v - w$ is orthog to W.

Observations: If W has basis $\{w_1, \dots, w_n\}$, then z is orthog to W iff z is orthog to each basis vector.

$$w = c_1 w_1 + \cdots + c_n w_n$$

so $0 = \langle z, w \rangle = c_1 \langle z, w_1 \rangle + \cdots + c_n \langle z, w_n \rangle$ for all c_i's

iff $\langle z, w_i \rangle = 0, \ i = 1, \dots, n.$



Theorem. Let $\{u_1, \dots, u_n\}$ be an orthonormal basis of W. Then the orthog projection of v onto W is

$$w = c_1 u_1 + \cdots + c_n u_n \qquad \text{where } c_i = \langle v, u_i \rangle$$

ie $$\boxed{w = \langle v, u_1 \rangle u_1 + \cdots + \langle v, u_n \rangle u_n}$$

If the basis $\{v_1, \dots, v_n\}$ is just orthogonal, then

$$\boxed{w = \frac{\langle v, v_1 \rangle}{\|v_1\|^2} v_1 + \cdots + \frac{\langle v, v_n \rangle}{\|v_n\|^2} v_n.}$$

Pf. Let orthog proj be

$$w = c_1 u_1 + \cdots + c_n u_n.$$

Then $$0 = \langle z, u_i \rangle = \langle v, u_i \rangle - \langle c_1 u_1 + \cdots + c_n u_n, u_i \rangle$$

$$= \langle v, u_i \rangle - c_1 \langle u_1, u_i \rangle - \cdots - c_n \langle u_n, u_i \rangle$$

only the i^{th} one is non-zero and equals 1

$$= \langle v, u_i \rangle - c_i \checkmark$$

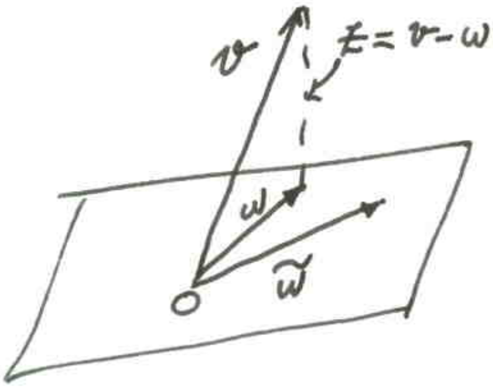
Example. Let $W = \operatorname{span}\{v_1, v_2\} \subset \mathbb{R}^3$,

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{with Euclidean inner prod.}$$

Then, since orthogonal, $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$w = \frac{\langle v, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle v, v_2 \rangle}{\|v_2\|^2} v_2 = \frac{1}{6} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Theorem Given $v \in V$, the closest point or least squares minimizer $w \in W$ to v is the same as the orthogonal projection of v onto W.



Why?

$$\|v - \tilde{w}\|^2 = \|z + (w - \tilde{w})\|^2$$

$$= \|w - \tilde{w}\|^2 + \cancel{2\langle w - \tilde{w}, z\rangle}^{\,=0} + \|z\|^2$$

So $\|v - \tilde{w}\|^2$ is minimized when $\tilde{w} = w$. Done.

Example Find the closest point in the span of v_1, v_2 to $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$p(x) = p(x_1, x_2) = \left\| \begin{pmatrix} 1 & 1 \\ -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\|^2$$

$=$ a quadratic f'n

$$= x^T K x - 2 x^T f + c$$

So solve $Kx = f$ for the minimizer x^*
(normal eq'ns)

Answer (from before) $w = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$.

Data Fitting Again!

$$A^TA\,x = A^Ty \qquad \text{Fit } y = \alpha + \beta t$$

$$\begin{pmatrix} 1 & \bar{t} \\ \bar{t} & \overline{t^2} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \bar{y} \\ \overline{ty} \end{pmatrix}$$

↑ invert, etc.

Fit $y = p(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n$

$$\Rightarrow \quad \alpha_0 \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{\underline{t}_0} + \alpha_1 \underbrace{\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}}_{\underline{t}_1} + \cdots + \alpha_n \underbrace{\begin{pmatrix} t_1^n \\ t_2^n \\ \vdots \\ t_m^n \end{pmatrix}}_{\underline{t}_n} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}}_{\underline{y}}$$

Thus we have a closest point problem!

Things would be easy if $\{\underline{t}_0, \underline{t}_1, \dots, \underline{t}_n\}$ were replaced by an orthog set, say $\{\underline{q}_0, \underline{q}_1, \dots, \underline{q}_n\}$. These are

Use $\langle u, v\rangle = \frac{1}{m}\sum_{i=1}^m u_i v_i = \overline{uv}$, $\langle \underline{t}_k, \underline{t}_l\rangle = \frac{1}{m}\sum_{i=1}^m t_i^k t_i^l = \overline{t^{k+l}}$

$q_0(t) = 1 \qquad \underline{q}_0 = \underline{t}_0, \ \|q_0\|^2 = 1$

$q_1(t) = t \qquad \underline{q}_1 = \underline{t}_1 - \frac{\langle t_1, q_0\rangle}{\|q_0\|^2} q_0 = t_1, \ \|q_1\|^2 = \overline{t^2}$

$\overline{t^3} = 0$

$q_2(t) = t^2 - \overline{t^2} \qquad \underline{q}_2 = \underline{t}_2 - \frac{\langle \underline{t}_2, \underline{q}_0\rangle}{\|\underline{q}_0\|^2}\underline{q}_0 - \frac{\langle \underline{t}_2, \underline{q}_1\rangle}{\|\underline{q}_1\|^2}\underline{q}_1$

$q_3(t) = t^3 - \frac{\overline{t^4}}{\overline{t^2}}t \qquad = \underline{t}_2 - \overline{t^2}\,\underline{t}_0, \ \|q_2\|^2 = \overline{t^4} - (\overline{t^2})^2$

$\vdots$

$$\underline{q}_3 = \underline{t}_3 - \frac{\overline{t^4}}{\overline{t^2}}\underline{t}_1, \ \|q_3\|^2 = \overline{t^6} - \frac{(\overline{t^4})^2}{\overline{t^2}}, \dots$$

Assume the t_i's are equally spaced and if t_i is there, so is $-t_i$

Then $\bar{t} = 0$ i.e. $-3, -2, -1, 0, 1, 2, 3$

$\overline{t^3} = 0$

$\vdots$

Then the least squares approx'n is

$p(t) = a_0 q_0(t) + a_1 q_1(t) + \cdots + a_n q_n(t)$

where

$$a_k = \frac{\langle \underset{\sim}{q_k}, \underset{\sim}{y} \rangle}{\| \underset{\sim}{q_k} \|^2} = \frac{\overline{q_k y}}{\overline{q_k^2}} \quad , k = 1, 2, \cdots, n.$$

Can carry this out to any degree without changing previously computed coeffs.

Review Example 5.42.