Math 415 Lecture 25
Section 5.6 Orthogonal Subspaces

Def'n Two subspaces $W, Z \subset V$ are orthogonal if every vector in W is orthogonal to every vector in Z.

It is enough to verify this on spanning sets (or bases) $w_1, \dots, w_k$ for W and $z_1, \dots, z_\ell$ for Z:

$$\langle w_i, z_j \rangle = 0 \text{ for all } i, j.$$

Example $W = \left\{ \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix} \middle| t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{w_1} \right\}$

$Z = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{z_1}, \underbrace{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}_{z_2} \right\}$

$w_1 \cdot z_1 = -2 + 2 + 0 = 0$, $w_1 \cdot z_2 = -3 + 0 + 3 = 0$. So $W \perp Z$

Def'n The orthog complement of a subspace $W \subset V$ is denoted by $W^\perp$ and consists of all vector orthog to W.

Why is it the case that $W \cap W^\perp = \{0\}$. Why is it that $W^\perp$ is a subspace? Note: $W^\perp$ will be different for different inner products.

Example: For W above, what is $W^\perp$? We need $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W^\perp$ if $\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\rangle = 0$. Use Euclidean i.p. $\Rightarrow$

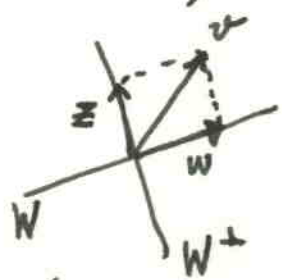
$x + 2y + 3z = 0$ Homog linear system:

$x = -2y - 3z$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y - 3z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow W^\perp = \operatorname{span}\{w_1, w_2\}$.

Theorem Let $W \subset V$ be a finite dimensional subspace of an inner product space. Then every $v \in V$ can be uniquely decompose into $v = w + z$ where $w \in W$, $z \in W^{\perp}$.

(Illustrate geometrically!)



(planar, Euclidean)

Proof. Let w be the orthogonal projection of v on W and set $z = v - w$.

closest point or use orthog basis

Uniqueness: If $v = w + z = w^* + z^*$, then $w - w^* = z^* - z$
$\Rightarrow$ $w - w^*$ and $z^* - z$ are in both W and $W^{\perp}$. Hence each is 0.

Example Use W and $W^{\perp}$ previously and $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
w is the orthog proj on W, z is the orthog proj on $W^{\perp}$ and each $=$ $v -$ other. Which is easiest? ...

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad w = \frac{\langle v, w_1 \rangle}{\| w_1 \|^2} w_1 = \frac{1}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

$$z = v - w = \frac{1}{14}\begin{pmatrix} 13 \\ -2 \\ -3 \end{pmatrix}$$

Review Example 5.52 on transparency.

Proposition: If W is finite dimensional, then $(W^{\perp})^{\perp} = W$.

EXAMPLE 5.52

Let $W \subset \mathbb{R}^4$ be the two-dimensional subspace spanned by the orthogonal vectors $\mathbf{w}_1 = (1, 1, 0, 1)^T$ and $\mathbf{w}_2 = (1, 1, 1, -2)^T$. Its orthogonal complement $W^\perp$ (with respect to the Euclidean dot product) is the set of all vectors $\mathbf{v} = (x, y, z, w)^T$ that satisfy the linear system

$$\mathbf{v} \cdot \mathbf{w}_1 = x + y + w = 0, \qquad \mathbf{v} \cdot \mathbf{w}_2 = x + y + z - 2w = 0.$$

Applying the usual algorithm—the free variables are y and w—we find that the solution space is spanned by

$$\mathbf{z}_1 = (-1, 1, 0, 0)^T, \qquad \mathbf{z}_2 = (-1, 0, 3, 1)^T,$$

which form a non-orthogonal basis for $W^\perp$. An orthogonal basis

$$\mathbf{y}_1 = \mathbf{z}_1 = (-1, 1, 0, 0)^T, \qquad \mathbf{y}_2 = \mathbf{z}_2 - \tfrac{1}{2}\mathbf{z}_1 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, 3, 1\right)^T,$$

for $W^\perp$ is obtained by a single Gram–Schmidt step. To decompose the vector $\mathbf{v} = (1, 0, 0, 0)^T = \mathbf{w} + \mathbf{z}$, say, we compute the two orthogonal projections:

$$\mathbf{w} = \tfrac{1}{3}\mathbf{w}_1 + \tfrac{1}{7}\mathbf{w}_2 = \left(\tfrac{10}{21}, \tfrac{10}{21}, \tfrac{1}{7}, \tfrac{1}{21}\right)^T \in W,$$
$$\mathbf{z} = \mathbf{v} - \mathbf{w} = -\tfrac{1}{2}\mathbf{y}_1 - \tfrac{1}{21}\mathbf{y}_2 = \left(\tfrac{11}{21}, -\tfrac{10}{21}, -\tfrac{1}{7}, -\tfrac{1}{21}\right)^T \in W^\perp.$$

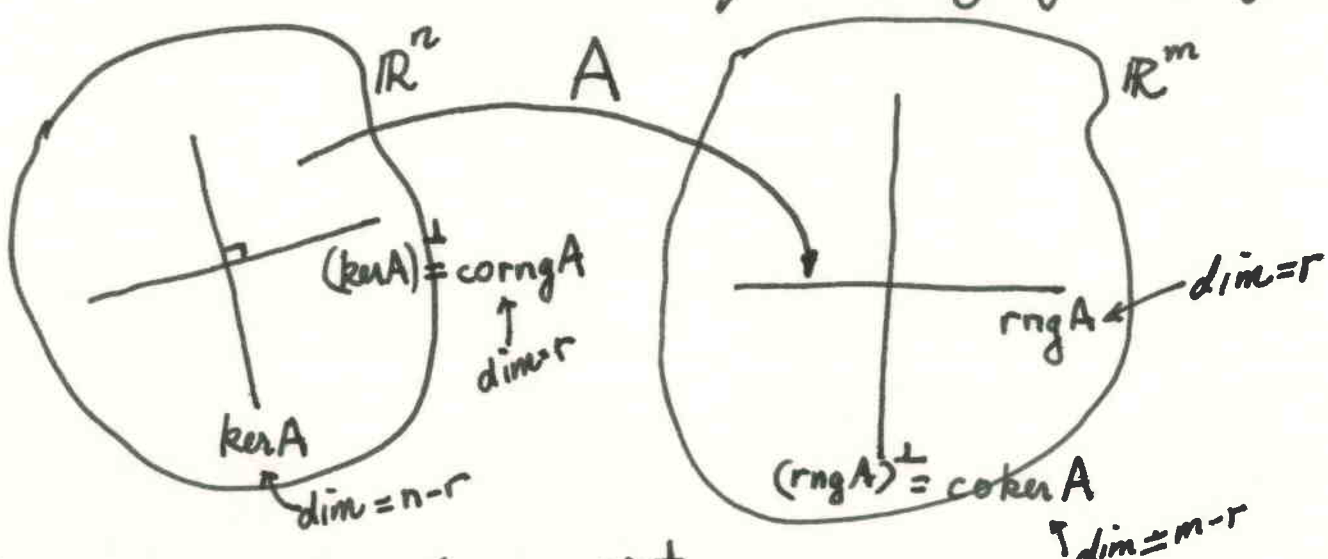
Orthogonality of the Fundamental Subspaces

$$Ax = b, \quad A \text{ is } m \times n, \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^m$$

For this system to have a sol'n, b must lie in rng A. If solutions are not unique then

$$x = x^* + z, \quad Ax^* = b, \quad z \in \ker A, \text{ ie } Az = 0.$$

Think of $y = Ax$ as defining a map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then we have the following geometry:



Theorem:
$$\ker A = (\text{corng} A)^\perp$$
$$(\text{rng} A)^\perp = \text{coker} A$$

$$0 = Az = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} z = \begin{pmatrix} u_1 \cdot z \\ \vdots \\ u_m \cdot z \end{pmatrix} \Rightarrow z \text{ is } \perp \text{ to all rows of } A$$
$$\Rightarrow z \text{ is } \perp \text{ to corng } A$$

A in terms of its rows

Apply the same argument now to A^T to get the second statement.

Theorem: (Fredholm Alternative) $Ax = b$ has a sol'n iff b is $\perp$ to the cokernel of A.

$Ax=b$ has a sol'n iff $y \cdot b = 0$ for all sol'ns of $\underbrace{A^T y = 0}_{\text{adjoint system}}$

Let $y_1, \dots, y_{m-r}$ be ~~the~~ a basis for coker A, then

$$\boxed{Ax=b \text{ has a sol'n iff } y_i \cdot b = 0,\ i=1,\dots,m-r}$$

<u>Example</u> $A = \begin{pmatrix} 1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 1 & -2 & -2 & 6 \end{pmatrix}$. What restrictions on b ensure that $Ax=b$ has a sol'n:

$$\left(\begin{array}{cccc|c} 1 & -3 & -7 & 9 & b_1 \\ 0 & 1 & 5 & -3 & b_2 \\ 1 & -2 & -2 & 6 & b_3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -3 & -7 & 9 & b_1 \\ 0 & 1 & 5 & -3 & b_2 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right)$$

So we need $b_3 - b_2 - b_1 = 0$

What is the cokernel of A?

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ -7 & 5 & -2 \\ 9 & -3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \\ 0 & -3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x = -z$
$y = -z$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}_{y}$

$$y \cdot b = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b_3 - b_2 - b_1 = 0$$