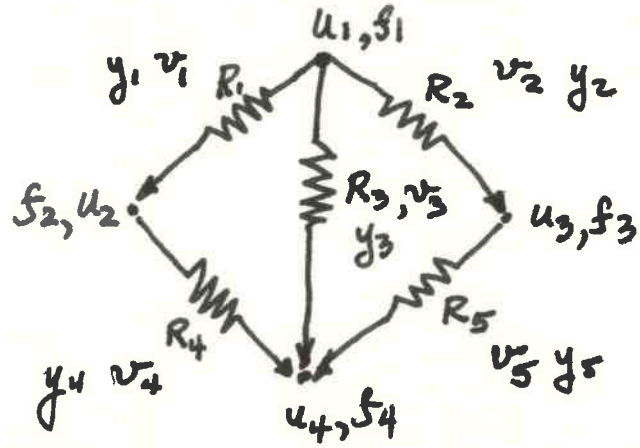
Math 415 Lecture 26

Sec 6.2 Electrical Networks (in equilibrium)



u_i's are the voltage potentials at each node

R_i's are the resistances of the wires

The voltage v_i in each wire is the difference in voltage potentials

Let y_i denote the current in the i^{th} wire

Let f_i denote the external current entering each node.

$$\left.\begin{aligned} v_1 &= u_1 - u_2 \\ v_2 &= u_1 - u_3 \\ v_3 &= u_1 - u_4 \\ v_4 &= u_2 - u_4 \\ v_5 &= u_3 - u_4 \end{aligned}\right\} \text{Write in matrix form}$$

$$v = Au$$

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \text{incident matrix of the network}$$

We need to interconnect y, v, R_i's and u_i's.

<u>Kirchhof's Law of Voltage</u> The sum of the ~~current~~ voltages around each closed circuit is zero

$$v_1 + v_4 - v_3 = 0, \quad v_2 + v_5 - v_3 = 0$$

ie $(1\ 0\ -1\ 1\ 0) \cdot v = 0, \quad (0\ 1\ -1\ 0\ 1) \cdot v = 0$

Question: Given a set of voltages v_i in the wires, can

we find the corresponding voltage potentials?
For this to be the case we need $v \in \operatorname{rng} A$. By Fredholm's criterion this is true iff v is orthog to $\operatorname{coker} A$. But these are the circuits, so Kirchhoff's Voltage Law ensures this! (Topography of the Network)

Ohm's Law (What is the constitutive content of Network)
($V = RI$ - for a fixed voltage, there is more current in lower resistance wires)

Here $v_k = R_k y_k$

Resistance Matrix

or $v = Ry$, $R = \operatorname{diag}(R_1, \dots, R_5)$

or $y = Cv$, $C = \operatorname{diag}(c_1, \dots, c_5)$

$= \operatorname{diag}\left(\frac{1}{R_1}, \dots, \frac{1}{R_5}\right)$

$=$ conductance matrix.

<u>Kirchhoff's Current Law</u> The net current leaving a node equals the external current comming into a node.

Example: If a 1 amp current is sent into node 1 and no other external currents, then $f_1 = 1$, $f_2 = f_3 = f_4 = 0$

So $y_1 + y_2 + y_3 = 1$, $-y_1 + y_4 = 0$, $-y_2 + y_5 = 0$, $-y_3 - y_4 - y_5 = 0$

$(1\,1\,1\,0\,0) \cdot y = 1$, $(-1\,0\,0\,1\,0) \cdot y = 0$

$(0\,-1\,0\,0\,1) \cdot y = 0$, $(0\,0\,-1\,-1\,-1) \cdot y = 0$

ie $\boxed{A^T y = f}$

So we have $v = Au$, $y = Cv$, $f = A^T y$

or

$$Ku = f \text{ where } K = A^T C A$$ ← Gram Matrix! = resistivity matrix of the network.

In our case

$$K = \begin{pmatrix} c_1+c_2+c_3 & -c_1 & -c_2 & -c_3 \\ -c_1 & c_1+c_4 & 0 & -c_4 \\ -c_2 & 0 & c_2+c_5 & -c_5 \\ -c_3 & -c_4 & -c_5 & c_3+c_4+c_5 \end{pmatrix}$$

If $R_i = 1$ for all i, then

$Ku = f$ becomes $$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(K \mid f) \to (\text{Gaussian Elim}) \to \left(\begin{array}{cccc|c} 3 & -1 & -1 & -1 & 1 \\ 0 & 5/3 & -1/3 & -4/3 & 1/3 \\ 0 & 0 & \frac{8}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Wow! Explain in terms of "excess current cannot lead to equilibrium".

So what f's are acceptable? K is singular! Why? Relate to fact that $\ker A$ has dimension 1.

$\ker A = \text{span}\left\{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\}$. Thus $\ker K = \text{span}\left\{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\}$ $= \text{coker} K$ (since K is sym)

Hence f must be $\perp$ to $\text{coker} K$, ie

$(1\,1 \cdots 1) \cdot f = 0$ or $f_1 + \cdots + f_5 = 0$ Makes Sense!

Example Suppose $k_i = 1 \; \forall i$ and $f = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$.

Then by Gaussian Elimination we find that

$$u = \begin{pmatrix} \frac{1}{2} + t \\ \frac{1}{4} + t \\ \frac{1}{4} + t \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1/4 \\ 1/4 \\ 0 \end{pmatrix} + t \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{is this expected?}}$$

How can we eliminate the ambiguity?
Minimum norm sol'n is one possibility.
Here we consider, instead, "grounding" one note, ie $u_i = 0$ for some i. If we choose $u_4 = 0$, the above shows that $t = 0$ so the other u_i's are determined.
But we could find this via a reduced problem. Eliminate the i^{th} column from A and the i^{th} row and column from K and i^{th} entry of f.