Math 415 Lecture 27 Sec 6.2 continued

Example. Set

$$A^* = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (fourth column has been eliminated)

$\Rightarrow$ lin indep columns. Set

$$K^* = (A^*)^T C (A^*) = \begin{pmatrix} c_1+c_2+c_3 & -c_1 & -c_2 \\ -c_1 & c_1+c_4 & 0 \\ -c_2 & 0 & c_2+c_5 \end{pmatrix}$$

$$f^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ (fourth entry elim.)

$$= \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$, now +ve definite

$$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad u_1 = \tfrac{1}{2}, \; u_2 = \tfrac{1}{4}, \; u_3 = \tfrac{1}{4}, \; \underbrace{u_4 = 0}_{\text{by design.}}$$

Batteries. Batteries are voltage sources along the wires.

b_k = voltage of the battery on wire k

$$v_k = u_i - u_j + b_k$$

or $\quad v = Au + b \quad , \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ = battery vector

Recall that we still have $y = Cv$ and $f = A^T y$.
If $f = 0$ we have a circuit driven only by batteries and in this case $A^T y = 0$, ie

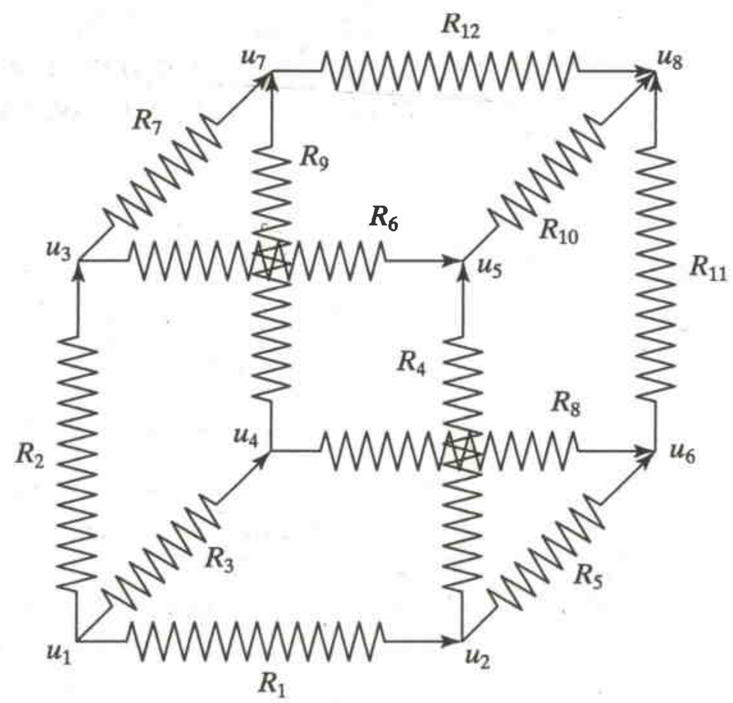
$$A^T C (Au + b) = 0$$

or
$$\underbrace{A^T C A}_{K} u = -A^T C b$$

or
$$Ku = -A^T C b \qquad (*)$$

If we set $\|v\| = \sqrt{v^T C v}$, a weighted innerproduct, then $(*)$ are just the normal eq'ns for the least squares solution of $Au = -b$.

EXAMPLE 6.4



Consider an electrical network running along the sides of a cube, where each wire contains a 2 ohm resistor and there is a 9 volt battery source on one wire. The problem is to determine how much current flows through the wire directly opposite the battery. Orienting the wires and numbering them as indicated in Figure 6.6, the incidence matrix is

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We connect the battery along wire 1 and measure the resulting current along wire 12. To avoid the ambiguity in the voltage potentials, we ground the last node and erase the final column from A to obtain the reduced incidence matrix $A^\star$. Since the resistance matrix R has all 2's along the diagonal, the conductance matrix is $C = \frac{1}{2}\,\mathrm{I}$. Therefore, the network resistivity matrix is

$$K^\star = (A^\star)^T C A^\star = \tfrac{1}{2}\,(A^\star)^T A^\star = \frac{1}{2}\begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 3 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 3 \end{pmatrix}.$$

The reduced current source vector corresponding to the battery

$$\mathbf{b} = (9,0,0,0,0,0,0,0,0,0,0,0)^T$$

along the first wire is

$$\mathbf{f}^\star = -\,(A^\star)^T C\,\mathbf{b} = \left(-\tfrac{9}{2}, \tfrac{9}{2}, 0, 0, 0, 0, 0\right)^T.$$

Solving the resulting linear system $K^\star \mathbf{u}^\star = \mathbf{f}^\star$ by Gaussian Elimination yields the voltage potentials

$$\mathbf{u}^\star = \left(-3, \tfrac{9}{4}, -\tfrac{9}{8}, -\tfrac{9}{8}, \tfrac{3}{8}, \tfrac{3}{8}, -\tfrac{3}{4}\right)^T.$$

Thus, the induced currents along the sides of the cube are

$$\begin{aligned} \mathbf{y} = C\,\mathbf{v} &= C\,(A^\star \mathbf{u}^\star + \mathbf{b}) \\ &= \left(\tfrac{15}{8}, -\tfrac{15}{16}, -\tfrac{15}{16}, \tfrac{15}{16}, \tfrac{15}{16}, -\tfrac{3}{4}, -\tfrac{3}{16}, -\tfrac{3}{4}, -\tfrac{3}{16}, \tfrac{3}{16}, \tfrac{3}{16}, -\tfrac{3}{8}\right)^T. \end{aligned}$$

In particular, the current on the wire that is opposite the battery is $y_{12} = -\frac{3}{8}$, flowing in the opposite direction to its orientation. The largest current flows through the battery wire, while wires 7, 9, 10 and 11 transmit the least. ●