Math 415 Lecture 28

Chapter 7 Linear Functions

Vectors in $\mathbb{R}^n$ generalize to functions in a f'n space
Matrix mult'n in $\mathbb{R}^n$ generalizes to ?

Def'n Let V, W be real vector spaces. A function $L : V \to W$ is called linear if

$$L[v_1 + w_2] = L[v] + L[w_2], \quad L[cv] = c\,L[v]$$

generalize to linear combos

for any vectors v_1 and, v_2 and v and any real number c.

Note: $L[0] = 0$ (take $c=0$ above), so origin is mapped to origin.
If $L : \mathbb{R} \to \mathbb{R}$, then $L[x] = ax$ for some const a.

$\lceil$ $L[x] = L[x \cdot 1] = x\,L[1] = ax$ where $a = L[1]$ $\rfloor$

If A is $m \times n$, then $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $\underline{L[v] = Av}$ is linear. Conversely

A linear f'n represented by a Matrix.

Theorem Every linear f'n $L : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by matrix mult'n by some Matrix.

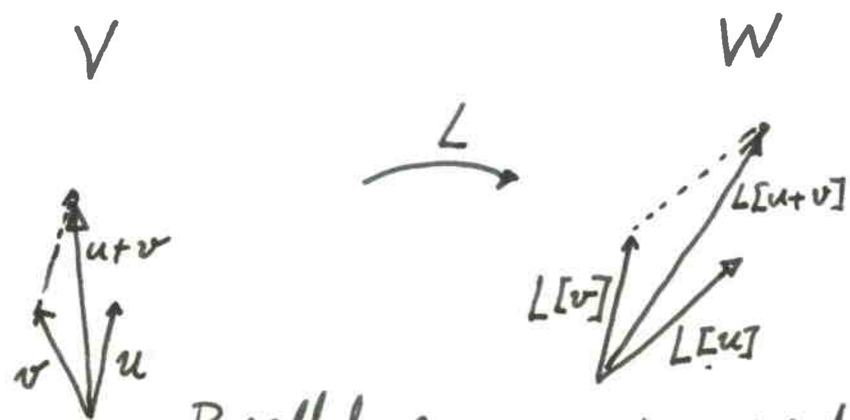
Pf. Let $e_1, \dots, e_n$ be the standard basis in $\mathbb{R}^n$, set $a_i = L(e_i)$ and $A = (a_1 \cdots a_n)$. Then

$$L[v] = L[v_1 e_1 + \cdots + v_n e_n] = v_1 L[e_1] + \cdots + v_n L[e_n]$$
$$= v_1 a_1 + \cdots + v_n a_n = \underline{Av}.$$

Sec 7.1 Linear Functions

Geometric Interpretations :

$$L[u+v] = L[u] + L[v]$$

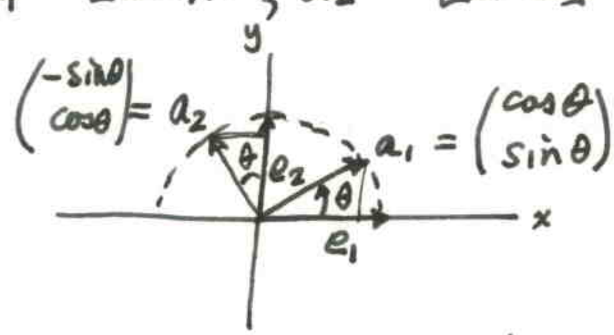


Parallelograms are preserved

Scalings are preserved

Example Rotation through angle θ counterclockwise $\to R_\theta$

$a_1 = L[e_1]$, $a_2 = L[e_2]$

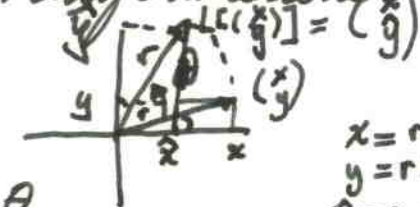


$$\Rightarrow A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \underbrace{R_\theta[v]}_{= \hat{v} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}} = Av = A\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{aligned} \hat{x} &= \cos\theta\, x - \sin\theta\, y \\ \hat{y} &= \sin\theta\, x + \cos\theta\, y \end{aligned}$$

7.2 Linear Transformations

Here we will talk mostly about linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^3$.



$x = r\cos\varphi$
$y = r\sin\varphi$
$\hat{x} = r\cos(\varphi+\theta)$
$\hat{y} = r\sin(\varphi+\theta)$
$\Rightarrow$ expand.

Rotations R_θ counterclockwise angle θ

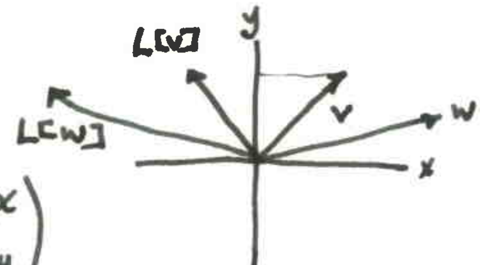
$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = R_\theta\left[\begin{pmatrix}x\\y\end{pmatrix}\right] = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{pmatrix}$$

$$= Q\begin{pmatrix}x\\y\end{pmatrix}$$

where Q is proper orthogonal, ie $Q^TQ = I$, $\det Q = +1$

Reflections $L\left[\begin{pmatrix}x\\y\end{pmatrix}\right] = \begin{pmatrix}-x\\y\end{pmatrix}$

reflection in y axis



$$= \underbrace{\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}}_{\text{A matrix representing a reflection}}\begin{pmatrix}x\\y\end{pmatrix}$$

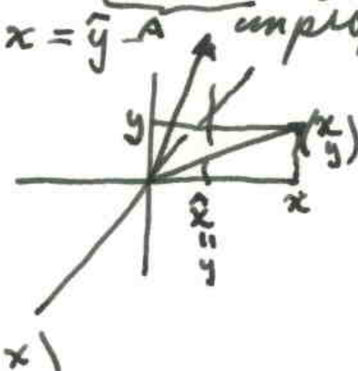
improper orthogonal

Reflection about x axis:

$$L\left[\begin{pmatrix}x\\y\end{pmatrix}\right] = \begin{pmatrix}x\\-y\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{A}\begin{pmatrix}x\\y\end{pmatrix}$$

improper orthogonal

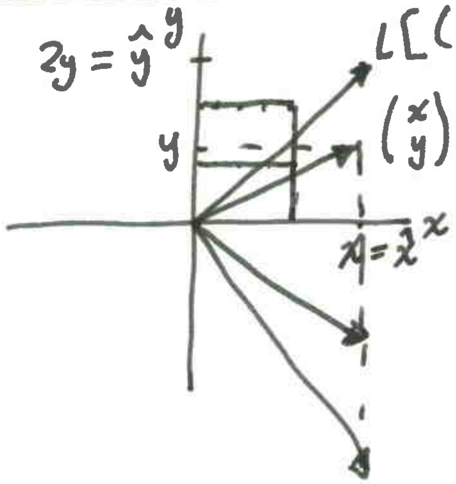
Reflection in line $x=y$



$$L\left[\begin{pmatrix}x\\y\end{pmatrix}\right] = \begin{pmatrix}y\\x\end{pmatrix}$$

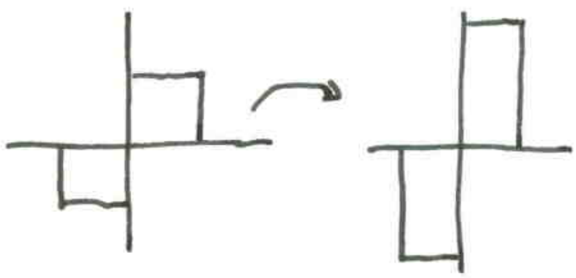
$$= \underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{A = \text{permult'n and improp orthogonal}}\begin{pmatrix}x\\y\end{pmatrix}$$

Stretches Double vectors in y direction:

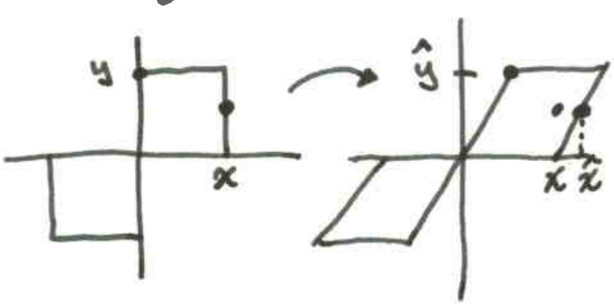


$$L\left[\begin{pmatrix}x\\y\end{pmatrix}\right] = \begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}x\\ 2y\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}}_{A}\begin{pmatrix}x\\y\end{pmatrix}$$

Best here to work in terms of boxes rather than just vectors



Shears Try $$L\left[\begin{pmatrix}x\\y\end{pmatrix}\right] = \begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}x+2y\\ y\end{pmatrix} = \overbrace{\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}}^{A\ =\ \text{elementary row op'n}}\begin{pmatrix}x\\y\end{pmatrix}$$



$\hat{y} = y$

$\hat{x} = x + 2y$