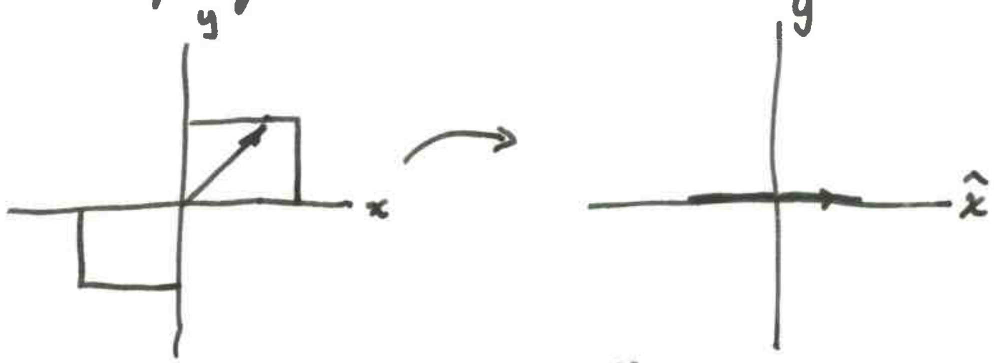
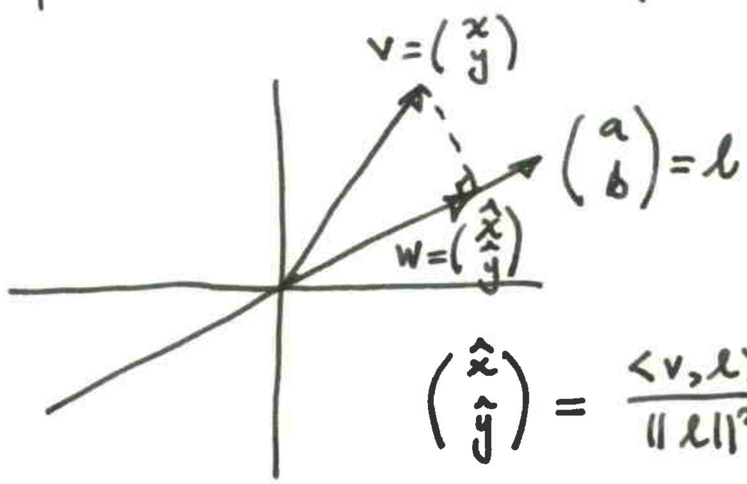
Math 415 Lecture 29 Sec 7.2 continued

Also note that all the transf'ns we have talked about to this point are invertible.
Look at

$$L[v] = L\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{A rank 1}} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is a projection onto the x-axis.





$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \frac{\langle v, \ell \rangle}{\|\ell\|^2} \ell$$

$$= \frac{ax + by}{a^2 + b^2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} a \dfrac{ax+by}{a^2+b^2} \\ b \dfrac{ax+by}{a^2+b^2} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \dfrac{a^2}{a^2+b^2} & \dfrac{ab}{a^2+b^2} \\ \dfrac{ab}{a^2+b^2} & \dfrac{b^2}{a^2+b^2} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

Note: $\det A = 0$ so rank 1.

Composition

Lemma: If V, W, Z are vector spaces and $L: V \to W$ and $M: W \to Z$ are linear f'ns, then $M \circ L: V \to Z$ is a linear f'n.

Pf

$$(M \circ L)(cv_1 + dv_2) = M[L[cv_1 + dv_2]]$$
$$= M[cL[v_1] + dL[v_2]]$$
$$= cM[L[v_1]] + dM[L[v_2]]$$
$$= c(M \circ L)[v_1] + d(M \circ L)[v_2]$$

Done

If $L: R^n \to R^m$ is represented by matrix A and $M: R^m \to \mathbb{R}^\ell$ is represented by matrix B, what matrix represents $M \circ L$?

$L[v] = Av$, $M[w] = Bw$, so

$$M \circ L[v] = M[L[v]] = M[Av] = (BA)v$$

indeed, this is where our def'n of matrix mult'n comes from!

Example $R_\varphi \circ R_\theta = R_{\varphi+\theta}$

ie

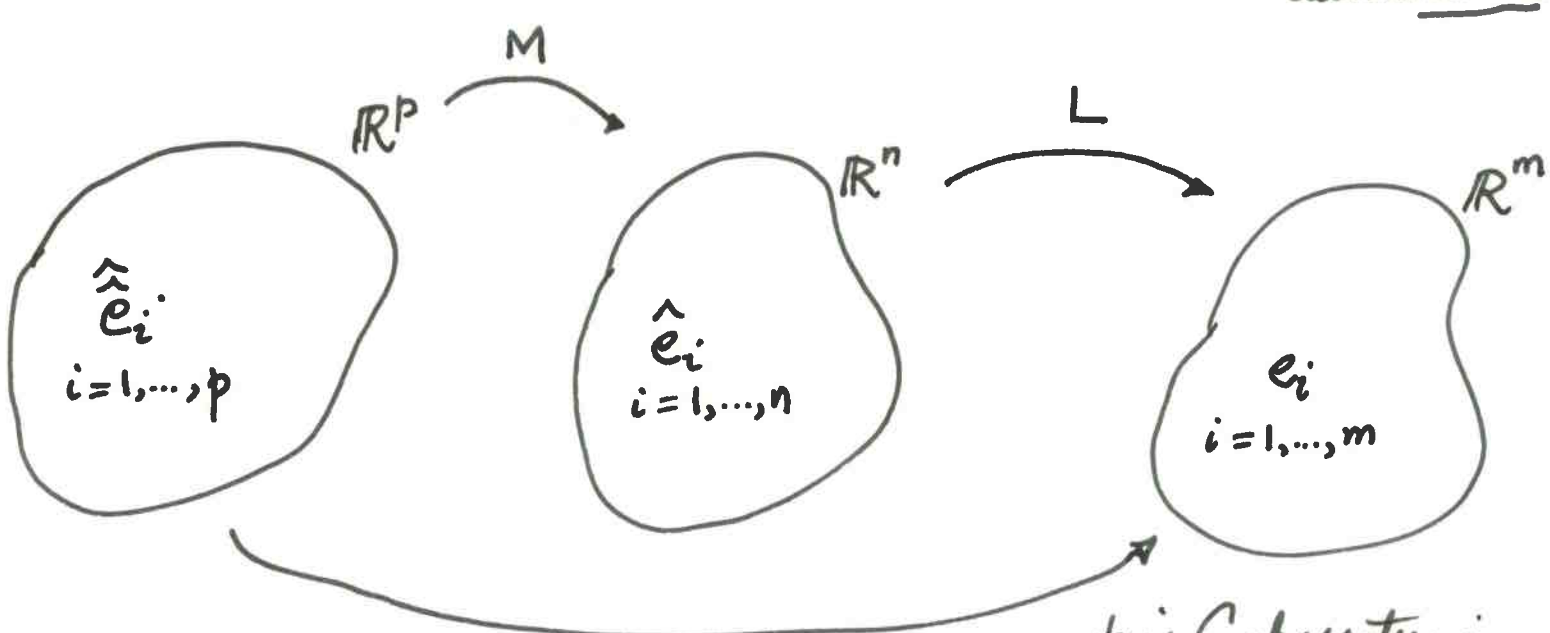
$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{pmatrix}$$

$$\begin{pmatrix} \cos\varphi\cos\theta - \sin\varphi\sin\theta & * \\ \sin\varphi\cos\theta + \cos\varphi\sin\theta & * \end{pmatrix} \Rightarrow \text{sum angle formulae.}$$

$$M[\hat{\hat{e}}_j] = b_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} = \sum_{k=1}^{n} \underbrace{b_{kj}}_{\text{wow, another matrix } B!} \hat{e}_k$$

$$L[\hat{e}_j] = a_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} = \sum_{i=1}^{m} a_{ij} e_i$$

Wow, an $m \times n$ array characterizing L. Let's call it a <u>matrix</u>! A



$$L \circ M[\hat{\hat{e}}_j] = c_j = \begin{pmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{pmatrix}^{L \circ M} = \sum_{i=1}^{m} c_{ij} e_i$$

matrix C characterizing $L \circ M$

BUT:

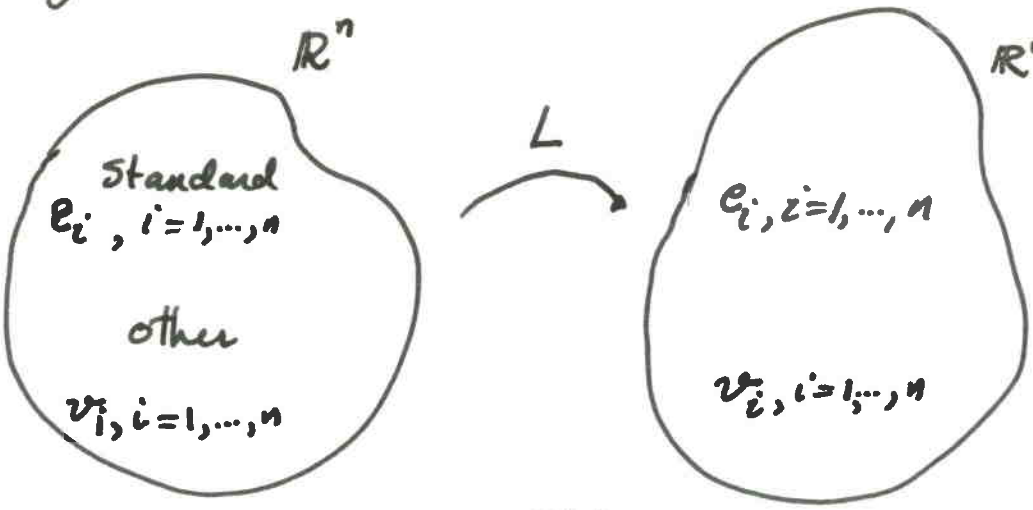
$$\| $$

$$L[M[\hat{\hat{e}}_j]] = L\Big[\sum_{k=1}^{n} b_{kj} \hat{e}_k\Big] = \sum_{k=1}^{n} b_{kj} L[\hat{e}_k] = \sum_{k=1}^{n} b_{kj} \sum_{i=1}^{m} a_{ik} e_i$$

$$= \sum_{i=1}^{m} \Big(\underbrace{\sum_{k=1}^{n} a_{ik} b_{kj}}\Big) e_i$$

So C comes from A and B!
Call this the product AB!!

Change of Basis ($\mathbb{R}^n \to \mathbb{R}^n$ only)



$v = x_1 e_1 + \dots + x_n e_n = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ coords rel to standard basis

$= y_1 v_1 + \dots + y_n v_n$, y_i's are coords of x rel to other basis

$= \underbrace{\begin{pmatrix} v_1 \cdots v_n \end{pmatrix}}_{S} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$S \Rightarrow S$ is non-singular $n \times n$ matrix (why?)

ie $Sy = x$ or equiv $y = S^{-1}x$

$L[e_j] = a_j = \sum_{i=1}^{n} a_{ij} e_i$ $\quad A = (a_{ij})$ matrix of L rel. to $\{e_i\}$

$L[v_j] = b_j = \sum_{i=1}^{n} b_{ij} v_i$ $\quad B = (b_{ij})$ matrix of L rel to $\{v_i\}$

We have connected x and y. Can we connect A and B?

Ax ← coords of $L[x]$ rel to $\{e_i\}$

$S^{-1}Ax$ ← coords of $L[x]$ rel to $\{v_i\}$

$By = (S^{-1}AS)y$ so $B = S^{-1}AS$. We call A and B "similar" matrices $\Rightarrow$ rep same L.

Example $L[v] = L\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] = \begin{pmatrix} x_1 - x_2 \\ 2x_1 + 4x_2 \end{pmatrix}$

$$= A\begin{pmatrix} x \\ x_2 \end{pmatrix} \text{ where } A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}.$$

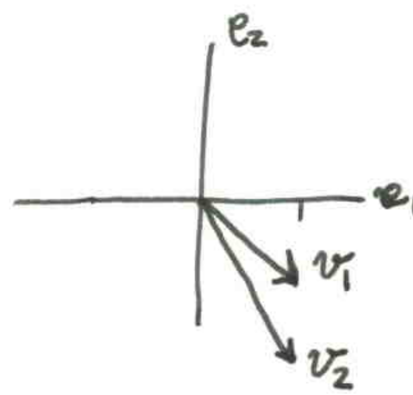
or: $L[e_1] = L\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e_1 + 2e_2$

$L[e_2] = L\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 4 \end{pmatrix} = -e_1 + 4e_2$.

Instead consider the basis

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= e_1 - e_2 \qquad\qquad = e_1 - 2e_2$$



$$S = \left(\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\right)$$

$$L[v_1] = L\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2v_1 = 2v_1 + 0v_2$$

$$L[v_2] = L\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right] = \begin{pmatrix} 3 \\ -6 \end{pmatrix} = 3v_2 = \underbrace{0v_1 + 3v_2}$$

$$B = \left(\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\right)$$

$$S^{-1}AS = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \checkmark$$

write $v_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $v_2 = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

$$Av_1 = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2v_1$$

i.e. $S\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right)$

$$Bv_1 = \left(\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\right)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = 2v_1$$