Math 415
Lecture 3
Sec 1.4

Recall from last day:

Elementary row operation of type 1.

Regular square matrix – (upper triangular, non-zero pivots)

$$\begin{aligned} 2y+z&=2\\ 2x+6y+z&=7\\ x+y+4z&=3\end{aligned} \quad\to\quad \left(\begin{array}{ccc|c} \overset{\text{problem!}}{0} & 2 & 1 & 2\\ 2 & 6 & 1 & 7\\ 1 & 1 & 4 & 3\end{array}\right)$$

$$\Updownarrow$$

$$\begin{aligned} 2x+6y+z&=7\\ 2y+z&=2\\ x+y+4z&=3\end{aligned} \quad\to\quad \left(\begin{array}{ccc|c} \overset{\text{ok, go!}}{2} & 6 & 1 & 7\\ 0 & 2 & 1 & 2\\ 1 & 1 & 4 & 3\end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 6 & 1 & 7\\ 0 & 2 & 1 & 2\\ 0 & 0 & \frac{9}{2} & \frac{3}{2}\end{array}\right) \text{ now back substitute}$$

Elementary row operation of type 2 – interchange two rows.

Def'n A square matrix is non-singular if it can be reduced to upper triangular form by row operations of type 1 and type 2 with non-zero entries – pivots – on the diagonal.

Thm $Ax=b$ has a unique soln for every vector b iff A is square and non-singular (Why?)

What does the LU factorization look like in this case?

Def'n A special permutation matrix is a matrix that interchanges two rows of a matrix (via matrix mult'n). A permutation matrix is a product of special ones.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ every row and every colum has a 1 and all other 0s

Note that $PP = P^2 = I$ (why?) ie P is its own inverse.

Let's use some special notation

$E_{ij}(a)$ = elementary matrix that adds $\underline{a}$ time row i to row j

Its inverse is $E_{ij}(-a)$ why?

Also let

P_{ij} = special permutation matrix that interchanges rows i and j.

Example

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & -1 \\ -3 & -5 & 6 & 1 \\ -1 & 2 & 8 & -2 \end{pmatrix}$$

$$E_{14}(1)E_{13}(3)E_{12}(-2)A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 4 & 7 & -2 \end{pmatrix}$$ "pivot"

$$P_{23}E_{14}(1)E_{13}(3)E_{12}(-2)A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 4 & 7 & -2 \end{pmatrix}$$

$$E_{24}(-4)P_{23}E_{14}(1)E_{13}(3)E_{12}(-2)A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -5 & -6 \end{pmatrix}$$ "pivot"

$$P_{34}E_{24}(-4)P_{23}E_{14}(1)E_{13}(3)E_{12}(-2)A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -1 \end{pmatrix} = U$$

Permuted LU factorization : $PA = LU$

$$P_{34}E_{24}(-4)\underbrace{P_{23}E_{14}(1)}_{E_{14}(1)P_{23}}E_{13}(3)E_{12}(-2)A = U$$

$$E_{14}(1)\underbrace{P_{23}\,E_{13}(3)}_{E_{12}(3)P_{23}}$$

$$E_{12}(3)\underbrace{P_{23}\,E_{12}(-2)}_{E_{13}(-2)P_{23}}$$

$$\underbrace{P_{34}\,E_{24}(-4)}_{E_{23}(-4)P_{34}}E_{14}(1)\,E_{12}(3)\,E_{13}(-2)\,P_{23}A = U$$

$$E_{23}(-4)\underbrace{P_{34}\,E_{14}(1)}_{E_{13}(1)P_{34}}$$

$$E_{13}(1)\underbrace{P_{34}\,E_{12}(3)\,E_{13}(-2)}_{E_{12}(3)\,E_{14}(-2)\,P_{34}}$$

$$E_{23}(-4)\,E_{13}(1)\,E_{12}(3)\,E_{14}(-2)\,\overbrace{P_{34}P_{23}}^{P}A = U$$

$$PA = \underbrace{E_{14}(2)E_{12}(-3)E_{13}(-1)E_{23}(4)}_{L}U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

Alternate Formulation (see pg 29)

$$A:\quad A \sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & -1 \\ -3 & -5 & 6 & 1 \\ -1 & 2 & 8 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 4 & 7 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 4 & 7 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -5 & -6 \end{pmatrix} \sim \overset{U}{\overset{\shortparallel}{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -1 \end{pmatrix}}}$$

$$L:\quad I \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} = L$$

$$P:\quad I \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = P$$

$$Ax = b$$

$$PAx = Pb = \hat{b}$$

$$LUx = \hat{b}$$

$$\underbrace{Ux = c}_{\text{Backward Subs}^{n}} \text{ provided } \underbrace{Lc = \hat{b}}_{\text{Forward Subs}^{n}}$$

Variant:

$$PA = \underset{\text{special lower diag.}}{L}\ \underset{\text{diagonal}}{D}\ \overset{\text{special upper diag}}{V}$$

$$U = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}^{D} \overbrace{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & \frac{6}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{V}$$