Math 415 Lecture 30 Sec 7.2 continued

Inverses

Let $L : V \to W$ be a linear function. If $M : W \to V$ is a function for which

$$L \circ M = I_W \quad , \quad M \circ L = I_V$$

explain these

then we say M is the inverse of L and write M as L^{-1}.
i.e. $L[M[w]] = w \;\forall w \in W, \quad M[L[v]] = v \;\forall v \in V.$

Proposition: The inverse operator is unique

Proposition: If $M = L^{-1}$, then $L = M^{-1}$, and so $(L^{-1})^{-1} = L$

Proposition: The inverse of a linear f'n, if it exists, is also a linear f'n.

More important: If L is rep. by matrix A
M is " " " B
then $M = L^{-1} \iff B = A^{-1}$.

Example:

$$v \qquad L : \mathbb{R}^4 \to M_{2\times 2}$$

$$L\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right] = \begin{pmatrix} v_1 & v_1+v_2 \\ v_3 & v_4 \end{pmatrix}$$

$\mathbb{R}^4$: e_1, e_2, e_3, e_4 standard

$M_{2\times2}$: $\hat{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\hat{e}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\hat{e}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\hat{e}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$L[e_1] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \hat{e}_1 + \hat{e}_2$$

$$L[e_2] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \hat{e}_2$$

$$L[e_3] = \hat{e}_3$$

$$L[e_4] = \hat{e}_4$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L^{-1}[\hat{e}_1] = e_1 - e_2$$

$$L^{-1}[\hat{e}_2] = e_2$$

$$L^{-1}[\hat{e}_3] = e_3$$

$$L^{-1}[\hat{e}_4] = e_4$$

So

$$\hat{v}$$

$$L^{-1}\left[\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}\right] = v_1\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + v_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + v_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 - v_1 \\ v_3 \\ v_4 \end{pmatrix}$$

$$L^{-1}[L[v]] = v \quad , \quad L[L^{-1}[\hat{v}]] = \hat{v}$$

Example (Non-matrix)(Operators)

$V = C^0[a,b]$, $W = \mathbb{R}$

a) $L[f] = \int_a^b f(x)\,dx$

$$L[cf+dg] = \int_a^b (c f(x) + d g(x))\,dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$$
$$= c L[f] + d L[g].$$

$V = C^0[a,b]$, $W = C^0[a,b]$

b) $M_c[f] = cf$ ie $M_c[f](x) = c f(x)$

scalar mult'n by c.

explain!

c) $V = C^1[a,b]$, $W = C^0[a,b]$

$D[f] = f'$ ie $D[f](x) = f'(x)$.

Is this linear?

Is there an inverse?

Undo differentiation

$\Rightarrow$ integration!

But what about "+C"

Def'n $\mathcal{L}(V,W)$ denotes the set of all linear functions $L: V \to W$. By our theorem, $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ can be identified with $M_{m\times n}$, the set of $m\times n$ matrices. If $n = m$ we talk about a linear transformation. The vectors in the domain and target spaces have the same size, ie look at R_θ.

d) $V = \{f \in C^1[a,b] \mid f(a) = 0\}$, $W = C^0[a,b]$

$D[f] = f'$. Does D have an inverse $J: W \to V$?

Yes, $J[g](x) = \int_a^x g(s)\,ds$ ie

$$J[D[f]](x) \stackrel{\circ}{=} \int_a^x D[f](s)\, ds$$

$$= \int_a^x f'(s)\,ds = f(x) - f(a) = f(x)$$

$$D[J[g]]x = \frac{d}{dx}\left(\int_a^x g(s)\,ds\right) = g(x)$$

ie $J \circ D$ = identity in V

$D \circ J$ = identity in W.