Math 415 Lecture 31

Chapter 8 Eigenvalues + Eigenvectors

Def'n Let A be $n \times n$. A scalar λ is called an eigenvalue of A if there is a non-zero vector $v \neq 0$, called an eigenvector, such that

$$Av = \lambda v$$

"eigen" is German for "proper" or "characteristic"

Note: $v = 0$ is always a sol'n. We want non-zero sol'ns. Rewrite as

$$(A - \lambda I)v = 0$$

$\Rightarrow A - \lambda I$ is singular

So $\det(A - \lambda I) = 0$ Characteristic Eq'n.

Example $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 - 1$$
$$= \lambda^2 - 6\lambda + 8$$
$$= (\lambda - 4)(\lambda - 2) = 0$$

$\Rightarrow \lambda = 4, \lambda = 2$

Case 1: $\lambda = 4$ $\quad A - 4I = \begin{pmatrix} 3-4 & 1 \\ 1 & 3-4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ is this singular? YES.

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} v = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -a + b = 0$ (only one eq'n) ie $a = b$, so

$v = \begin{pmatrix} a \\ a \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, do we expect an arbit a?

So $\lambda = 4$ is an eigenvalue with eigenvector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Case 2: $\lambda = 2$

$$A - 2I = \begin{pmatrix} 3-2 & 1 \\ 1 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{is it singular?}$$

$$(A-2I)v = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $a+b=0 \Rightarrow v = \begin{pmatrix} a \\ -a \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So $\lambda = 2$ is an eigenvalue with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

<u>Example</u>

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$(A-\lambda I) = \begin{pmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} \begin{matrix} -\lambda & -1 \\ 1 & 2-\lambda \\ 1 & 1 \end{matrix}$$

$$= \dots$$

$$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = \underbrace{-(\lambda-1)^2(\lambda-2)}_{\text{explain about guesswork!}} = 0$$

$\lambda = 1,\ \lambda = 2$

Case $\lambda = 1$:

$$(A-I)v = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow a+b+c=0$$

$a = -b-c$

$$v = \begin{pmatrix} -b-c \\ b \\ c \end{pmatrix} = b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{A Basis!!}$$

Case $\lambda = 2$:

$$(A - 2I)v = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

. . . . . .

$a + c = 0$, $c = -a$
$a + b = 0$ $b = -a$

$$\text{so } v = \begin{pmatrix} a \\ -a \\ -a \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \lambda_1 = 1, \quad v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \hat{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ \lambda_2 = 2, \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \end{array}}$$

An eigenvalue λ is <u>simple</u> if it is a single root of the characteristic eqn. Here $\lambda_2 = 2$ is simple and $\lambda_1 = 1$ is a <u>double eigenvalue</u>

Set $V_\lambda = \ker(A - \lambda I)$ = eigenspace corresp to eigenvalue λ.

$$V_1 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

↑ a subspace!

$$V_2 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

<u>Example</u> $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$, $0 = \det(A - \lambda I)$

$= -(\lambda+1)^2(\lambda-3)$

$\Rightarrow \lambda = -1$ is double, $\lambda = 3$ is simple

$$\boxed{\begin{array}{l} \lambda_1 = -1, \quad v_1 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \\ \lambda_2 = 3, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \end{array}}$$

A is <u>incomplete</u> if it does not have a full set (of 3) eigenvectors.