Example

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$0 = \det(A - \lambda I) = -(\lambda+1)(\lambda^2 - 2\lambda + 5)$$

$\Rightarrow$

$$\lambda_1 = -1 \qquad v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1+2i \qquad v_2 = \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 1-2i \qquad v_3 = \begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}$$

Prop'n If $\lambda = \mu + i\nu$ is a complex eigenvalue for a real matrix A with corresp eigenvector $v = x + iy$, then $\bar{\lambda} = \mu - i\nu$ is also an eigenvalue and its eigenvector is $\bar{v} = x - iy$.

Pf $Av = \lambda v \Rightarrow \bar{A}\bar{v} = \bar{\lambda}\bar{v} \Rightarrow A\bar{v} = \bar{\lambda}\bar{v}$ done.

Properties of Eigenvalues:

$$p_A(\lambda) = \det(A - \lambda I) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0.$$

characteristic polynomial of A

Look at 2×2 case:

$$p_A(\lambda) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - (a+d)\lambda + ad - bc$$
$$= \lambda^2 - (\operatorname{tr} A)\lambda + \det A.$$

In general we can show that

$$c_n = (-1)^n, \quad c_{n-1} = (-1)^{n-1} \operatorname{tr} A, \quad c_0 = \det A$$

Let the eigenvalues be $\lambda_1, \dots, \lambda_n$ (possibly with repetitions) Then

$$\begin{aligned} P_A(\lambda) &= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \\ &= (-1)^n \lambda^n + (-1)^{n-1} \lambda^{n-1} (\lambda_1 + \cdots + \lambda_n) + \cdots + \lambda_1 \cdots \lambda_n \end{aligned}$$