Math 415 Lecture 34

As long as the eigenvectors are linearly independ there are n of them, S is invertible.

Section 8.4

Theorem 8.20 Let $A = A^T$ be a real sym $n \times n$ matrix. Then

a) all eigenvalues of A are real and complete
b) eigenvectors corresp to distinct eigenvalues are orthog (in dot product)
c) there is an orthonormal basis of $\mathbb{R}^n$ made up of eigenvectors of A.

Example $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$\lambda_1 = 4$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

orthog in dot product

$u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ orthonormal basis of $\mathbb{R}^2$

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow S^{-1}AS = \Lambda$$

$$\Leftrightarrow S^TAS = \Lambda$$

since S is an orthogonal matrix!

Here is a consequence:

<u>Thm</u> A sym matrix $K = K^T$ is +ve definite <u>iff</u> all its eigenvalues are strictly +ve

<u>Pf</u> Assume K is +ve definite and v is an eigenvector for eigenvalue λ. Then

$$x^T K x > 0 \qquad \forall x \neq 0,$$

so

$$\begin{aligned} v^T K v &> 0 \\ \| \\ v^T(\lambda v) \\ \| \\ \lambda \|v\|^2 \end{aligned}$$

so

$$\lambda > 0.$$

For a general x, use an orthonormal basis $u_1, \dots, u_n$ of eigenvectors to write

$$x = c_1 u_1 + \cdots + c_n u_n$$

Then

$$\begin{aligned} Kx &= c_1 K u_1 + \cdots + c_n K u_n \\ &= c_1 \lambda_1 u_1 + \cdots + c_n \lambda_n u_n . \end{aligned}$$

Then

$$\begin{aligned} x^T K x &= (c_1 u_1 + \cdots + c_n u_n) \cdot (c_1 \lambda_1 u_1 + \cdots + c_n \lambda_n u_n) \\ &= \lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2 > 0 \end{aligned}$$

$\Rightarrow$ K is +ve def.

<u>Example</u> $K = \begin{pmatrix} 8 & 0 & 1 \\ 0 & 8 & 1 \\ 1 & 1 & 7 \end{pmatrix} \Rightarrow -\lambda^3 + 23\lambda^2 - 174\lambda + 432 = 0$

$-(\lambda - 9)(\lambda - 8)(\lambda - 6) = 0$

so $\lambda = 9, 8, 6$ Done.

Pf of 8.20

a) If $A = A^T$ then $(Av)\cdot w = v\cdot(Aw)$ where $v\cdot w = v^T\bar{w}$ in $\mathbb{C}^n$.

Let λ be a complete with assoc eigenvector $v \in \mathbb{C}^n$. Then

$$(Av)\cdot v = (\lambda v)\cdot v = \lambda \|v\|^2$$

$$v\cdot(Av) = v\cdot(\lambda v) = \bar{\lambda}\|v\|^2 \text{ (Why?)}$$

$\therefore \lambda = \bar{\lambda}$, ie λ is real.

b) Let $Av = \lambda v$, $Aw = \mu w$ with $\lambda \neq \mu$.

Then

$$\lambda v\cdot w = (Av)\cdot w = v\cdot(Aw) = \bar{\mu}\, v\cdot w = \mu v\cdot w$$

$\Rightarrow (\lambda - \mu)\, v\cdot w = 0$

$\Downarrow \lambda \neq \mu$ so $v\cdot w = 0$.