Math 415 Lecture 35

The Spectral Theorem:

$$A = Q \Lambda Q^T$$

} Geometric Interpretation?

given that A is symmetric.
How does this come out of $A = S \Lambda S^{-1}$?

Completing the square (again):

$$\begin{aligned} q(x) = x^T A x &= x^T Q \Lambda Q^T x \\ &= (Q^T x)^T \Lambda (Q^T x) \\ &= y^T \Lambda y, \quad y = Q^T x \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \end{aligned}$$

Example

$$\begin{aligned} q(x) &= 3x_1^2 + 2x_1x_2 + 3x_2^2 \\ &= (x_1 \ x_2) \underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (y_1 \ y_2) \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= 4y_1^2 + 2y_2^2 \end{aligned}$$

$\lambda = 4$, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow u = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda = 2$, $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow u = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{pmatrix}$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{pmatrix}$$

Why sign change?

where $y = Q^T x$, ie

$$y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2)$$

$$y_2 = \frac{1}{\sqrt{2}}(-x_1 + x_2)$$

Optimization Principles

$\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$
where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

} what are the eigenvectors?

$$q(y) = y^T \Lambda y = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$
where $y_1^2 + \cdots + y_n^2 = 1.$

$$\begin{aligned} q(y) &\leq \lambda_1 y_1^2 + \lambda_1 y_2^2 + \cdots + \lambda_1 y_n^2 \\ &= \lambda_1 (y_1^2 + y_2^2 + \cdots + y_n^2) \\ &= \lambda_1 \end{aligned}$$

$$q(e_1) = \lambda_1$$

Thus
$$\lambda_1 = \max \{ q(y) \mid \|y\| = 1 \}.$$

But $\|y\| = \|Q^T x\| = \|x\|$, so

$$\boxed{\begin{array}{l} \lambda_1 = \text{largest eigenvalue} = \max \{ q(x) \mid \|x\| = 1 \}. \\ v_1 \text{ is where max is attained!} \end{array}}$$

Similarly,
$$\lambda_n = \text{smallest eigenvalue} = \min \{ q(x) \mid \|x\| = 1 \}.$$

Example: maximize $3x^2 + 2xy + 3y^2$ subj to $x^2 + y^2 = 1$.
Answer: 4. Min is 2!

$$\frac{q(x)}{\|x\|^2} = \frac{x^T A x}{\|x\|^2} = \left(\frac{x}{\|x\|}\right)^T A \left(\frac{x}{\|x\|}\right) \leq \lambda_1$$

provided $x \neq 0$.

Reighly-Ritz

ie $\lambda_1 = \text{largest eigenvalue} = \max\left\{ \frac{q(x)}{\|x\|^2} \;\middle|\; x = 0 \right\}$

$$2 \leq \frac{3x_1^2 + 2x_1x_2 + 3x_2^2}{x_1^2 + x_2^2} \leq 4$$

$$\begin{array}{ccccccc} \lambda_1 & \geq & \lambda_2 & \geq & \cdots & \geq & \lambda_n \\ \downarrow & & \downarrow & & & & \downarrow \\ v_1 & & v_2 & & \cdots & & v_n \end{array}$$

Then

$$\lambda_j = \max\left\{ q(x) \;\middle|\; \|x\| = 1,\ x \cdot v_1 = 0, \ldots, x \cdot v_{j-1} = 0 \right\}$$