Math 415
Lecture 4
Sec 1.5

Definition: X is the inverse of A if
$$XA = AX = I$$
Necessarily A and X must be square + same size. Set $X = A^{-1}$

Examples

① $$P_{ij} P_{ij} = I$$ for any special permutation matrix

② $$E_{ij}(a) E_{ij}(-a) = E_{ij}(-a) E_{ij}(a) = I.$$

Theorem (proof later) A has an inverse iff A is non-singular.

③ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$

$$AX = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} ax+bz = 1 \\ cx+dz = 0 \end{matrix} \qquad \begin{matrix} ay+bw = 0 \\ cy+dw = 1 \end{matrix}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$x = \frac{d}{ad-bc}, \; z = -\frac{c}{ad-bc} \qquad y = -\frac{b}{ad-bc}, \; w = \frac{a}{ad-bc}$$

so $$X = A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ provided $ad-bc \neq 0$.

Now verify that $XA = I$ too!

Set $\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$

This is the determinant of A. We see that A^{-1} exists iff $\det A \neq 0$. This will be a general property later, not just a 2×2 property.

Properties of Inverses:

a) the inverse is unique

$$XA = I = AX$$
$$YA = I = AY$$
$$\Rightarrow X\underbrace{AY}_{I} = IY = Y$$
$$X = Y \checkmark$$

b) $(A^{-1})^{-1} = A$

c) $(AB)^{-1} = B^{-1}A^{-1}$

Generalization: $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdot \cdots \cdot A_2^{-1} A_1^{-1}.$

Using an inverse to solve systems:

$$Ax = b$$
$$\underbrace{(A^{-1}A)}_{I}x = A^{-1}b$$
$$x = A^{-1}b.$$

Looks neat, but finding A^{-1} is generally more complicated than the LU factoring.

How would we find the inverse? Gauss-Jordan Elimination.

$$AX = A\overbrace{(x_1 | x_2 | x_3)}^{X \text{ written as 3 columns}} = I = (e_1 | e_2 | e_3)$$

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= (Ax_1 | Ax_2 | Ax_3)$$

ie. $Ax_1 = e_1$, $Ax_2 = e_2$, $Ax_3 = e_3$

Three linear systems to solve for the columns of $X = A^{-1}$.

$$\left(A \,\middle|\, \begin{matrix}1\\0\\0\end{matrix}\right), \quad \left(A \,\middle|\, \begin{matrix}0\\1\\0\end{matrix}\right), \quad \left(A \,\middle|\, \begin{matrix}0\\0\\1\end{matrix}\right)$$

Instead consider

$$\left(A \,\middle|\, \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right)$$

and do Gaussian Elimination to get to $\left(\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \,\middle|\, X\right)$

this is A^{-1}.