Math 415

Lecture 5

Sec 1.6 Transposes

If $A = (a_{ij})$, then A^T = transpose of A = (a_{ji})

ie the rows of A become the columns of A^T

If A is $m \times n$, then A^T is $n \times m$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$ "reflect across the diagonal".

Some properties:

a) $(A^T)^T = A$

b) $(A+B)^T = (a_{ij}+b_{ij})^T = (a_{ji}+b_{ji}) = (a_{ji}) + (b_{ji}) = A^T + B^T$

c) $(AB)^T = B^T A^T$

$$\left(\sum_{k=1}^n a_{ik} b_{kj}\right)^T = \left(\sum_{k=1}^n a_{jk} b_{ki}\right) = \left(\sum_{k=1}^n b^T_{ik} a^T_{kj}\right) = B^T A^T$$

d) $(A^{-1})^T = (A^T)^{-1}$

Pf Set $X = (A^{-1})^T$. We want to show $A^T X = X A^T = I$.

$A^T X = A^T (A^{-1})^T = (A^{-1} A)^T = I^T = I$

$X A^T = \dots$ similar $= I$ ✓

Definition: A square matrix A is symmetric if $A = A^T$.
For 3×3 this means

$$A = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}.$$

Such matrices come up often in practice.

Thm A symmetric matrix A is regular iff it has a factorization

$$A = \underset{\text{special lower diag}}{L}\ \underset{\text{diagonal.}}{D}\ L^T$$

Use Gaussian elim to get $A = LU$. Write U as DV and then set $L = V^T$.

$$LDV = A = A^T = (LDV)^T = V^T \underset{\text{already sym.}}{D} L^T$$

$$\Rightarrow L = V^T.$$

Example $$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{5}{2} \end{pmatrix}$$

Sec 1.8 General Matrices

Review Echelon form using the transparency (pg 60 in the text)
Note that

$$PA = LU$$

still applies if we assume U is in echelon form

We still use Gaussian elimination.
Review this with the example on transparencies

Definition: The *rank* of an $m \times n$ matrix is the number of pivots
Thus

$$\text{rank } A \leq \min(n, m)$$

Theorem A square $n \times n$ matrix is non-singular iff its rank is n.
Why is this trivial for us?

Rank is independent of echelon form! later)

$$Ax = b \Rightarrow (A \mid b) \Rightarrow \underbrace{(U \mid c)}_{\text{now how do we solve?}}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{array}\right), \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \leftarrow \text{satisfied!}$$

Def'n The variables corresponding to pivot columns are basic variables and all other variables are free variables.

We solve for the basic variables in terms of the free variables by back substitution.

Try it out in this example: $\underline{\underline{x}} + 2y + z = 3$

$2\underline{\underline{z}} = 4 \to z = 2$

$x = 3 - 2y - z = 1 - 2y$

arbitrary

An $m \times n$ matrix is said to be in *row echelon form* **if it has the following "staircase" structure:**

$$U = \begin{pmatrix} * & * & \cdots & * & * & \cdots & * & * & \cdots & \cdots & * & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & * & \cdots & * & * & \cdots & \cdots & * & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & * & \cdots & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$