Math 415

Lecture 7

Sec 2.1 Vector Spaces

Read Sections 2.1 & 2.2 (in advance if possible)

Much of the structure of linear algebra is designed to study linear problems. In this section we introduce the objects of study: vectors. These are general objects that live in vector spaces. Let us use V to denote a vector space and use v, w, etc to denote vectors.

In Chp 7 we introduce linear functions from one vector space to another. Let X and V be two vector spaces and $L : X \to V$ a linear function. Then we will be interested in linear eq'n of the type

(*) $L(x) = v \quad , x \in X, v \in V.$

Compare this to $Ax = b$. But * could be linear eq'ns, linear differential equations, linear integral eq'ns, etc, and we want to intro structures that will apply to them all.

For a vector space we need a set V and two operations, addition denoted $\oplus$ and scalar multiplication denoted $\circ$. Here are some examples:

Example 1 Spaces of n-tuples $\mathbb{R}^n$

$$v = \underbrace{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}_{\text{a vector in this space}}, \quad w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \quad \text{define} \quad v \oplus w = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}, \quad c \odot v = \begin{pmatrix} c v_1 \\ \vdots \\ c v_n \end{pmatrix}$$

addition of reals

scalar mult'n of reals

Example 2 Spaces of Matrices $M_{m\times n}$

$$M = \underbrace{\begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \\ m_{m1} & \cdots & m_{mn} \end{pmatrix}}_{\text{a vector in this space}}, \quad N = \begin{pmatrix} n_{11} & \cdots & n_{1n} \\ \vdots & & \vdots \\ n_{m1} & \cdots & n_{mn} \end{pmatrix}$$

define
$$M \oplus N = \begin{pmatrix} m_{11}+n_{11} & \cdots & m_{1n}+n_{1n} \\ \vdots & & \\ m_{m1}+n_{m1} & \cdots & m_{mn}+n_{mn} \end{pmatrix}$$
addition of reals

$$c \odot M = \begin{pmatrix} c m_{11} & \cdots & c m_{1n} \\ \vdots & & \\ c m_{m1} & \cdots & c m_{mn} \end{pmatrix}$$
scalar mult'n of reals

Example 3 Spaces of Polynomials

$$P^{(n)} = \{ p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \}$$ i.e. degree ≤ n

If $\underbrace{p(x) = a_n x^n + \cdots + a_1 x + a_0}_{\text{a vector in this space}}$ and $q(x) = b_n x^n + \cdots + b_1 x + b_0$

define

$$p(x) \oplus q(x) = (a_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0)$$

$$c \odot p(x) = (c a_n) x^n + \cdots + (c a_1) x + (c a_0).$$

add'n of reals

scalar mult'n of reals

Example 4 Function Spaces $\mathcal{F}(S)$ (real fns defined on S.

$C^0(\mathbb{R}) = \{ f(x) \mid f(x)$ is continuous real valued for all $x \in \mathbb{R}\}$

Define

addition of reals

$$(f \oplus g)(x) = f(x) + g(x) \quad \forall x$$

$$(c \odot f)(x) = c \cdot f(x) \quad \forall x.$$

~~addition~~ scalar mult of reals

We can talk about all 4 examples simultaneously in terms of a <u>Vector</u> <u>Space</u> V

V is a set equipped with two operations satisfying (Show transparency + talk about it!)

Question: Is the set of upper triangular matrices a vector sp?

" " " special " " " "

Example Let V be a vector space, with operations $\oplus$ and $\circ$, S a set of real numbers and define $\mathcal{F} = \{ f : S \to V \}$

Is $\mathcal{F}$ a vector space? We need operations!

$$(f \oplus g)(x) \overset{?}{=} f(x) + g(x) \quad \forall x \in S$$

+ in $\mathcal{F}$ + in V

$$(c \odot f)(x) \overset{?}{=} c \circ f(x)$$

scalar mult^n in $\mathcal{F}$ Scalar mult^n in V

Think about this in terms of functions from $x \in \mathbb{R}$ to 3x3 upper triang.

i.e. $f(x) = \begin{pmatrix} x & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^2 \end{pmatrix}$ is an element of $\mathcal{F}(\mathbb{R}$, 3x3 upper triangular)

Definition 2.1

A *vector space* is a set V equipped with two operations:

(i) *Addition*: adding any pair of vectors $\mathbf{v}, \mathbf{w} \in V$ gives another vector $\mathbf{v} + \mathbf{w} \in V$;

(ii) *Scalar Multiplication*: multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\,\mathbf{v} \in V$;

subject to the following axioms, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$:

(a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

(b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

(c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.

(d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$.

(e) *Distributivity*: $(c + d)\,\mathbf{v} = (c\,\mathbf{v}) + (d\,\mathbf{v})$, and $c\,(\mathbf{v} + \mathbf{w}) = (c\,\mathbf{v}) + (c\,\mathbf{w})$.

(f) *Associativity of Scalar Multiplication*: $c\,(d\,\mathbf{v}) = (c\,d)\,\mathbf{v}$.

(g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\,\mathbf{v} = \mathbf{v}$.