Math 415
Lecture 8
Sec 2.2 Subspaces

Def'n A subspace of a vector space V is a subset $W \subset V$ that is also a vector space (with the same operations).

Prop'n $W \subset V$ is a subspace of V if
a) $v, w \in W \Rightarrow v+w \in W$ (closed under addition)
b) $v \in W \Rightarrow cv \in W$ (closed under scalar mult^{n})

Equivalently: $v, w \in W \rightarrow av+bw \in W$ $\forall$ scalars a and b.

Examples ① $W = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} \subset \mathbb{R}^3$

$$v = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, w = \begin{pmatrix} z \\ u \\ 0 \end{pmatrix} \Rightarrow av+bw = a\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + b\begin{pmatrix} z \\ u \\ 0 \end{pmatrix} = \begin{pmatrix} ax+bz \\ ay+bu \\ 0 \end{pmatrix} \in W$$

② Set of solutions $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| 3x+2y-z=0 \right\} \subset \mathbb{R}^3$ (plane in $\mathbb{R}^3$)

$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, w = \begin{pmatrix} r \\ s \\ t \end{pmatrix}$ both in S implies

$$v+w = \begin{pmatrix} x+r \\ y+s \\ z+t \end{pmatrix}. \text{ But } 3(x+r)+2(y+s)-(z+t)$$
$$= (3x+2y-z) + (3r+2s-t)$$
$$= 0+0=0, \text{ so } v+w \in S.$$

Now try cv for yourself.

③ Span of two vectors. Set

$v_1 = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, $= \text{span}(v_1, v_2) = \{ a V_1 + bV_2 \mid a, b$ are real numbers$\}$

$$s_1 = a_1 V_1 + b_1 V_2, \ S_2 = a_2 V_1 + b_2 V_2 \Rightarrow aS_1 + bS_2 = a(a_1 V_1 + b_1 V_2) + b(a_2 V_1 + b_2 V_2)$$

$= (aa_1 + ba_2)\, v_1 + (ab_1 + bb_2)\, v_2 \in S.$ ✓

READ Examples 2.11 and 2.12

Note: you can show directly that the span of any collection of vectors is a subspace (see Sec 2.3 for definitions).

We already pointed out that the set of solutions of $Ax = 0$ for an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. We call this subspace the kernel of A, $\ker A$.

The two ways in which we characterize subspaces are:

a) as the kernel of a matrix

b) as the span of some vectors.

Indeed in Examples 2 and 3 above, S and W are the same thing, one expressed as a kernel (of $A = (3\ 2\ -1)$) and the other as the span of v_1 and v_2. To show this note that

$$3x + 2y - z = 0 \Leftrightarrow x = -\tfrac{2}{3}y + \tfrac{1}{3}z \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}y + \frac{1}{3}z \\ y \\ z \end{pmatrix}$$

$$= -\tfrac{1}{3}y \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \tfrac{1}{3}z \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \text{ and so } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ is in span}\{v_1, v_2\}$$

Sec 2.3 Span & Linear Independence

Def'n: Let $v_1, \dots, v_k$ lie in vector space V. Then

$$c_1 v_1 + \cdots + c_k v_k = \sum_{i=1}^{k} c_i v_i \qquad , c_1, \dots, c_n \text{ real no.s}$$

is called a <u>linear combination</u> of $v_1, \dots, v_k$. Also

$$\text{span}(v_1, \dots, v_k) = \text{set of all linear combinations of } v_1, \dots, v_k$$

Note: $\text{span}(v_1, \dots, v_k)$ is a subspace of V. (why?)

How do we know if a given vector is in the span of others? ie is

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \in \text{span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ +1 \end{pmatrix} \right) \ ?$$

error in text!

This means

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -3 \\ +1 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ -2c_1 - 3c_2 \\ c_1 + c_2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} c_1 + 2c_2 &= 0 \\ -2c_1 - 3c_2 &= 1 \\ c_1 + c_2 &= -1 \end{aligned} \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 1 & +1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$c_1 + 2c_2 = 0$$

$$c_2 = 1 \Rightarrow c_1 = -2$$

yes!

$$\underline{\text{Example}} \quad P^{(2)} = \{ p(x) = a_2 \overset{\substack{\text{in } P^{(2)} \\ \downarrow}}{x^2} + a_1 \overset{\substack{\text{in } P^{(2)} \\ \downarrow}}{x} + a_0 \cdot \overset{\substack{\text{in } P^{(2)} \\ \swarrow}}{1} \} = \operatorname{span}\{1, x, x^2\}$$

$$\underline{\text{Example}} \quad M_{2\times 2}:$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and so

$$M_{2\times 2} = \operatorname{span}\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{4 vectors in } M_{2\times 2}} \}$$