Math 415
Lecture 9
Sec 2.3 Continued

Example $T^{(2)}$ = space of quadratic trig polynomials

$$= \{ t(x) = a_0 + a_1\cos x + a_2 \sin x + a_3 \cos^2 x + a_4 \sin x \cos x + a_5 \sin^2 x \}$$

$$= \operatorname{span}\{1, \cos x, \sin x, \cos^2 x, \cos x \sin x, \sin^2 x\}$$

$$= \operatorname{span}\{1, \cos x, \sin x, \underset{\cos^2 x - \sin^2 x}{\underset{\shortparallel}{\cos 2x}}, \underset{2\sin x\cos x}{\underset{\shortparallel}{\sin 2x}}\}$$

So how few do we need to span a space?
(Read Example 2.17 in detail)

Def'n $v_1, \dots, v_k \in V$ are <u>linearly dependent</u> if a non-trivial linear combo of these is 0:

$$c_1 v_1 + \cdots + c_k v_k = 0 \text{ with } c_i \neq 0 \text{ for at least one } i$$

Otherwise we say $v_1, \dots, v_k$ are linearly independent

We can use dependence to eliminate one of the spanners + so get a smaller set still spanning our space

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ero vector

Examples ① $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$

so $\quad v_1 - 2v_2 + v_3 = 0$

But $c_1 V_1 + c_2 V_2$

$$= c_1 \begin{pmatrix}1\\2\\-1\end{pmatrix} + c_2 \begin{pmatrix}0\\3\\1\end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ 2c_1+3c_2 \\ -c_1+c_2 \end{pmatrix}$$

$$= \underset{\substack{\uparrow \\ \text{row reduce}}}{\begin{pmatrix}1 & 0\\2 & 3\\-1 & 1\end{pmatrix}} \begin{pmatrix}c_1\\c_2\end{pmatrix} \overset{?}{=} \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\to \begin{pmatrix}1 & 0\\0 & 3\\0 & 1\end{pmatrix} \to \begin{pmatrix}1 & 0\\0 & 3\\0 & 0\end{pmatrix} \Rightarrow c_1 = 0,\ 3c_2 = 0$$

$\Rightarrow$ *independent*

<u>Example</u> ② $1 - \cos^2 x - \sin^2 x = 0$ (an identity)

a nontrivial linear combo of $1, \cos x, \sin x, \cos^2 x, \cos x \sin x, \sin^2 x$.

$\Rightarrow$ these six are linearly dependent

Look at the example ① above. What we have shown is that

$$c_1 V_1 + \cdots + c_n V_n = Ac \quad \text{where } c = \begin{pmatrix}c_1\\ \vdots \\ c_n\end{pmatrix}$$

$$A = (V_1 \cdots V_n)$$

<u>Thm</u>. a) $V_1, \dots, V_k$ are linearly dependent if $Ac = 0$ has a non-trivial solution

b) $V_1, \dots, V_k$ are linearly indep iff $Ac = 0$ has only the trivial solution $c = 0$

c) b is in the span of $V_1, \dots, V_k$ if $Ac = b$ is <u>compatible</u>

Do example 2.22 via transparency.

Lemma: Any collection of $k > n$ vectors in $\mathbb{R}^n$ is linearly dependent, ie

$$Ac = 0$$

A has more columns than rows $\Rightarrow$ there is a non-trivial sol'n.

Prop'n: k vectors in $\mathbb{R}^n$ are linearly indep iff $A = (v_1, \dots, v_n)$ ($n \times k$ matrix with $k \leq n$) has rank k.

Continue example 2.22

EXAMPLE 2.22

Let us determine whether the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \tag{2.14}$$

are linearly independent or linearly dependent. We combine them as column vectors into a single matrix

$$A = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 0 & -4 & 2 \\ -1 & 4 & 6 & 3 \end{pmatrix}.$$

According to Theorem 2.21, we need to figure out whether there are any nontrivial solutions to the homogeneous equation $A\mathbf{c} = \mathbf{0}$; this can be done by reducing A to row echelon form

$$U = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & -6 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.15}$$

The general solution to the homogeneous system $A\mathbf{c} = \mathbf{0}$ is

$$\mathbf{c} = (2c_3 - c_4, -c_3 - c_4, c_3, c_4)^T,$$

where c_3, c_4—the free variables—are arbitrary. Any nonzero choice of c_3, c_4 will produce a nontrivial linear combination

$$(2c_3 - c_4)\mathbf{v}_1 + (-c_3 - c_4)\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

that adds up to the zero vector. We conclude that the vectors (2.14) are linearly dependent.

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Let us now see which vectors $\mathbf{b} \in \mathbb{R}^3$ lie in the span of the vectors (2.14). Accord
to Theorem 2.21, this will be the case if and only if the linear system $A\mathbf{c} = \mathbf{b}$
a solution. Since the resulting row echelon form (2.15) has a row of all zeros, th
will be a compatibility condition on the entries of $\mathbf{b}$, and hence not every vector
in the span. To find the precise condition, we augment the coefficient matrix,
apply the same row operations, leading to the reduced augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 4 & b_1 \\ 0 & -6 & -6 & -6 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + \frac{7}{6}b_2 - \frac{4}{3}b_1 \end{array}\right).$$

Therefore, $\mathbf{b} = (b_1, b_2, b_3)^T$ lies in the span if and only if $-\frac{4}{3}b_1 + \frac{7}{6}b_2 + b_3 =$
Thus, these four vectors only span a plane in $\mathbb{R}^3$.