Probability Worksheet # 2

1. Evaluate the following quantities.

   (a) 5! $= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

   (b) $\frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = 8 \cdot 7 = 56$

   (c) $4! + 3! = 4 \cdot 3 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 1 = 24 + 6 = 30$

   (d) $(4+3)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

   (e) $0! = 1$

2. Suppose that two fair six-sided dice (one red and one blue) are tossed. Let A denote the event that the sum is 6, let B denote the event that the sum is 7, and let C denote the event that the red die is a 4.

   (a) Are events A and C independent?

   $P(A) = \frac{5}{36}$ $(1+5, 2+4, 3+3, 4+2, 5+1)$

   $P(C) = 1/6$ since $P(A \cap C) \neq P(A) \cdot P(C)$

   $P(A \cap C) = 1/36$ A and C are not independent

   (b) Are events B and C independent?

   $P(B) = 6/36$ $(1+6, 2+5, 3+4, 4+3, 5+2, 6+1)$

   $P(C) = 1/6$ since $P(B \cap C) = P(B) \cdot P(C)$

   $P(B \cap C) = 1/36$ B and C are independent

3. Tom and Sarah each flip a coin. If they each get tails then Tom pays Sarah \$1. If they each get heads then Tom pays Sarah \$5. If their coin flips do not match then Sarah pays Tom \$2.50. Is this a fair game? If not then who can expect to benefit the most from this game and what are the expected winnings per game?

   expected winnings per game for Sarah are $\underbrace{\tfrac{1}{4}}_{P(TT)}(\$1) + \underbrace{\tfrac{1}{4}}_{P(HH)}(\$5) + \underbrace{\tfrac{1}{2}}_{P(TH \text{ or } HT)}(-\$2.50) = \$0.25$ or 25¢

4. In how many different orders can the four letters A, B, C, D be written, no letter being repeated in any one arrangement?

$4 \cdot 3 \cdot 2 \cdot 1 = 24$

5. How many different license plates can be made if each plate has three letters followed by 4 digits? Answer the question twice – first if repetition is allowed and then if repetition is not allowed.

repetition allowed: $26^3 \cdot 10^4$

repetition not allowed: $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

6. (a) How many 4-digit positive integers are there? (No leading zeroes allowed.)

$9 \cdot 10 \cdot 10 \cdot 10 = 9000$

(b) How many 4-digit positive integers have distinct digits?

$9 \cdot 9 \cdot 8 \cdot 7 = 4536$

(c) How many 4-digit positive integers have only odd digits?

$5 \cdot 5 \cdot 5 \cdot 5 = 625$

(d) How many 4-digit positive integers are even?

$9 \cdot 10 \cdot 10 \cdot 5 = 4500$

(e) How many 4-digit positive integers are even and have distinct digits?

that end in 0 | $9 \cdot 8 \cdot 7 \cdot 1 = 504$

that end in 2, 4, 6, or 8 | $8 \cdot 8 \cdot 7 \cdot 4 = 1792$

TOTAL | $504 + 1792 = 2296$

7. A club has 20 members (8 men and 12 women).

(a) How many ways can the club elect 3 of its members where one will serve as president, one as vice-president, and one as treasurer?

$${}_{20}P_3 = 20 \cdot 19 \cdot 18 = 6840$$

(b) How many ways can the club choose 3 of its members to serve in equal capacity on a committee?

$${}_{20}C_3 = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$$

(c) How many ways can the club choose 4 of its members to serve in equal capacity on a committee if the committee must consist of precisely two men and two women?

$${}_8C_2 \times {}_{12}C_2 = \frac{8 \cdot 7}{2 \cdot 1} \cdot \frac{12 \cdot 11}{2 \cdot 1} = 28 \cdot 66 = 1848$$

(d) How many ways can the club choose 4 of its members to serve in equal capacity on a committee if the committee must consist of at least two women?

$$\underbrace{{}_8C_2 \times {}_{12}C_2}_{2M,\,2W} + \underbrace{{}_8C_1 \times {}_{12}C_3}_{1M,\,3W} + \underbrace{{}_8C_0 \times {}_{12}C_4}_{0M,\,4W}$$

$$\frac{8 \cdot 7}{2 \cdot 1} \cdot \frac{12 \cdot 11}{2 \cdot 1} + 8 \cdot \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} + 1 \cdot \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 1848 + 1760 + 495 = 4103$$

8. How many distinct positive divisors do each of the following numbers have?

(a) $2^5 \times 3^2 \times 5^4 \times 7$

each divisor is of form $2^a \cdot 3^b \cdot 5^c \cdot 7^d$

with $a \in \{0,1,2,3,4,5\}$, $b \in \{0,1,2\}$, $c \in \{0,1,2,3,4\}$, $d \in \{0,1\}$

this gives $6 \cdot 3 \cdot 5 \cdot 2 = 180$ distinct positive divisors.

(b) 3400 $= 2^3 \cdot 5^2 \cdot 17$

each divisor is of form $2^a \cdot 5^b \cdot 17^c$

with $a \in \{0,1,2,3\}$, $b \in \{0,1,2\}$, $c \in \{0,1\}$

this gives $4 \cdot 3 \cdot 2 = 24$ distinct positive divisors

9. In a standard 52-card deck, each of the 13 card values A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K occurs 4 times (once in each of the 4 suits $\spadesuit$, $\heartsuit$, $\clubsuit$, $\diamondsuit$). A hand is a subset of the deck. (Addendum: Poker players will note that in my definition of *straight* and *flush* in parts (c) and (d), I should not have included any hand that was a *straight flush* or a *royal flush*.)

(a) What is the total number of possible 5-card hands?

$${}_{52}C_5 = 2598960$$

(b) How many 5-card hands contain a full house? A full house consists of 3 cards with one common card value and 2 cards with another common card value.

$$\underbrace{({}_{13}C_1)}_{\text{choose triple value}} \cdot \underbrace{({}_4C_3)}_{\text{chose triple suits}} \cdot \underbrace{({}_{12}C_1)}_{\text{choose pair value}} \cdot \underbrace{({}_4C_2)}_{\text{choose pair suits}} = 3744$$

(c) How many 5-card hands contain a straight? A straight is 5 consecutive cards from the sequence A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A without regard to suit.

$$\underbrace{10}_{\text{choose starting value}} \cdot \underbrace{4^5}_{\text{chose suits}} = 10240$$

(however techically we should subtract the 40 straight and royal flushes)

(d) How many 5-card hands contain a flush? A flush is 5 cards all in the same suit.

$$\underbrace{(4)}_{\text{choose suit}}\underbrace{({}_{13}C_5)}_{\text{chose cards within suit}} = 5148$$

(however, techically we should subtract the 40 straight and royal flushes)

(e) What is the probability that a 5-card hand contains a full house? a straight? a flush?

just divide each ~~re~~ result in parts b, c and d by 2598960

10. Suppose that each of 7 people is asked to think of an integer between 1 and 20 (inclusive). What is the probability that at least two of them choose the same number?

$$1 - P(\text{no two think of same number})$$

$$= 1 - \frac{20}{20} \cdot \frac{19}{20} \cdot \frac{18}{20} \cdot \frac{17}{20} \cdot \frac{16}{20} \cdot \frac{15}{20} \cdot \frac{14}{20}$$

$$= 1 - \frac{390700800}{1280000000}$$

$$= \frac{889299200}{1280000000}$$

$$= 0.694765$$

11. Suppose that 23 people are chosen at random from a crowd. Show that the probability that at least two of them share the same birthday (just the day, not the day and year) is greater than 1/2. Assume no one was born on February 29th.

$$1 - P(\text{no two share same birthday})$$

$$= 1 - \frac{365 \cdot 364 \cdot 363 \cdots 344 \cdot 343}{(365)^{23}}$$

$$\approx 1 - 0.49$$

$$\approx 0.51 > \frac{1}{2}$$