Rational Numbers, Decimals, Geometric Series, Approximations

1. Circle each rational number.

   $\frac{-2}{5}$, $\frac{3}{-4}$, 0, $^-2$, 3, $\sqrt{2}$, 3.14, $\frac{22}{7}$, π

   $\frac{0}{1}$ $\frac{-2}{1}$ $\frac{3}{1}$ $\frac{314}{100}$

2. Rewrite the following quantities in decimal form.

   (a) $\frac{3}{100} + \frac{5}{10000}$ $= 0.0305$

   (b) $\frac{4}{10^3} + \frac{3}{10^4}$ $= 0.0043$

   (c) $10^6 \cdot \left(\frac{4}{10^4} + \frac{3}{10^9}\right)$ $= 4 \cdot 10^2 + 3 \cdot \frac{1}{10^3} = 400.003$

3. Find the decimal representation for the following numbers. If you obtain a repeating decimal, be sure to clearly show which digits repeat.

   $\frac{1}{8}$ $\frac{2}{3}$ $\frac{5}{6}$ $\frac{3}{16}$ $\frac{4}{9}$ $\frac{3}{80}$ $\frac{3}{7}$

   $\frac{1}{8} = 0.125$ $\frac{2}{3} = 0.\overline{6}$

   $\frac{5}{6} = 0.8\overline{3}$ $\frac{3}{16} = 0.1875$ $\frac{4}{9} = 0.\overline{4}$

   $\frac{3}{80} = 0.0375$ $\frac{3}{7} = 0.\overline{428571}$

4. The repeating decimal $0.45\overline{81} = 0.459818181\ldots$ has a period of length 2 since 81 repeats. One might be tempted to say that the period could also be 4 since 8181 repeats. However for the period we always look at the smallest repeating part. Without converting to decimal form, determine the maximum length of the period for the following fractions. It will be helpful to think about the possible remainders obtained when doing long division.

   (a) $\frac{5}{17}$ max length = 16 (actual length = 16)

   (b) $\frac{2}{37}$ max length = 36 (actual length = 3)

   (c) $\frac{6}{13}$ max length = 12 (actual length = 6)

5. Explain precisely when a fraction can be represented by a terminating decimal.

if when written in simplest terms, (lowest)

the only prime factors in the denominator are 2 or 5

6. Which of the following numbers can be represented by a terminating decimal. You should not have to determine the decimal form ~~in order~~ to answer this question.

$\frac{17}{80} \downarrow \frac{17}{2^4 \cdot 5}$ $\qquad$ $\frac{7}{48} \downarrow \frac{7}{2^4 \cdot 3}$ $\qquad$ $\frac{3^4 \cdot 21^6}{2^7 \cdot 5^3 \cdot 10^2} \downarrow \frac{3^{10} \cdot 7^6}{2^9 \cdot 5^5}$ $\qquad$ $\frac{19}{2^4 \cdot 5^2 \cdot 7^3}$ already in simplest terms $\qquad$ $\frac{81}{2^7 \cdot 3^2 \cdot 5^8} \downarrow \frac{3^2}{2^7 \cdot 5^8}$

7. What is the sum of the finite geometric series $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$? Are there any conditions which must be true in order for this sum to be valid?

$\frac{a - ar^{n+1}}{1-r}$ $\quad$ $(r \neq 1)$

8. What is the sum of the infinite geometric series $a + ar + ar^2 + ar^3 + \cdots$? Are there any conditions which must be true in order for this sum to be valid?

$\frac{a}{1-r}$ $\quad$ $(-1 < r < 1)$

9. Determine the sum of the following geometric series or explain why the formulas above are not applicable. You do not need to simplify your answers.

(a) $1024 + 512 + 256 + 128 + \cdots + 4 + 2 + 1$ $\quad r = \frac{1}{2}$

$\text{sum} = \frac{1024 - 1/2}{1 - 1/2} = 2047$

(b) $1 + 5^2 + 5^4 + 5^6 + \cdots + 5^{16}$

$r = 5^2 \quad \text{sum} = \frac{1 - 5^{18}}{1 - 5^2}$

(c) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

$r = \frac{1}{3} \quad \text{sum} = \frac{1}{1 - 1/3} = 3/2$

(d) $5 + 10 + 20 + 40 + \cdots$

$r = 2$, formula is not applicable. in this case the sum is infinite.

(e) $3+2+\frac{4}{3}+\frac{8}{9}+\cdots$

$r=\frac{2}{3}$ $\quad$ sum $=\frac{3}{1-2/3}=9$

(f) $\frac{2^2}{3}+\frac{2^4}{3^2}+\frac{2^6}{3^3}+\cdots$

$r=\frac{4}{3}$, formula is not applicable. in this case the sum is infinite

(g) $8-4+2-1+\frac{1}{2}-\frac{1}{4}+\cdots$

$r=-\frac{1}{2}$ $\quad$ sum $=\frac{8}{1-(-1/2)}=\frac{16}{3}$

(h) $1+0.4+0.16+0.064+\cdots$

$r=0.4$ $\quad$ sum $=\frac{1}{1-0.4}=\frac{5}{3}$

(i) $0.6+0.18+0.054+\cdots$

$r=0.3$ $\quad$ sum $=\frac{0.6}{1-0.3}=\frac{6}{7}$

10. Rewrite the following repeating decimals as infinite series. Compute their sums and write your answer as a simplified fraction.

(a) $0.\overline{4}=0.44444\ldots$ $=\frac{4}{10}+\frac{4}{100}+\frac{4}{1000}+\cdots$

$=\frac{4/10}{1-1/10}=\frac{4}{9}$

(b) $0.\overline{7}=0.77777\ldots$ $=\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\cdots$

$=\frac{7/10}{1-7/10}=\frac{7}{9}$

(c) $0.\overline{13}=0.13131313\ldots$ $=\frac{13}{100}+\frac{13}{10000}+\frac{13}{1000000}+\cdots$

$=\frac{13/100}{1-1/100}=\frac{13}{99}$

(d) $0.\overline{05}=0.05050505\ldots$ $=\frac{5}{100}+\frac{5}{10000}+\frac{5}{1000000}+\cdots$

$=\frac{5/100}{1-1/100}=\frac{5}{99}$

(e) $0.2\overline{1} = 0.21111111\ldots$ $= \frac{2}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots$

$= \frac{2}{10} + \frac{1/100}{1-1/10} = \frac{2}{10} + \frac{1}{90} = \frac{19}{90}$

(f) $0.0\overline{15} = 0.0151515\ldots$ $= \frac{15}{1000} + \frac{15}{100000} + \frac{15}{10000000} + \cdots$

$= \frac{15/1000}{1-1/100} = \frac{15}{990} = \frac{1}{66}$

11. A rubber ball rebounds to two thirds of the height from which it falls. If it is dropped from a height of 4 feet and is allowed to continue bouncing forever, what is the total distance it travels?

$4 + \underbrace{4\left(\frac{2}{3}\right) + 4\left(\frac{2}{3}\right)} + \underbrace{4\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right)^2} + \cdots$

$= 4 + 8\left(\frac{2}{3}\right) + 8\left(\frac{2}{3}\right)^2 + \cdots$

$= 4 + \frac{8\left(\frac{2}{3}\right)}{1-\frac{2}{3}} = 4 + 16 = 20\text{ ft}$

12. Approximate the following quantities without using a calculator.

(a) 3.2406×100.012 $\approx 3.24 \times 100 = 324$

(b) $56.43 \div 0.00103$ $\approx 56.43 \div 0.001$

$= 56.43 \times 1000 = 56430$

(c) 0.124×161 $\approx \frac{1}{8} \times 160 = 20$

(d) $13.813 + 840.199$ $\approx 13.8 + 840.2 = 854$

(e) $8 \div 0.249$ $\approx 8 \div \frac{1}{4} = 8 \times 4 = 32$

(f) $4.002 \times 2.013^2 \times 4.997^3$ $\approx 4 \times 2^2 \times 5^3$

$= 2000$