Math 172 **Test 1 Solutions** **Spring 2006**

1. (8 points) Consider the following discrete dynamical system.

$$a(n) = 3a(n-1) + 2$$

If $a(0) = 1$, then what is the value of $a(4)$?

$$a(1) = 3 \cdot a(0) + 2 = 3 \cdot 1 + 2 = 5$$
$$a(2) = 3 \cdot a(1) + 2 = 3 \cdot 5 + 2 = 17$$
$$a(3) = 3 \cdot a(2) + 2 = 3 \cdot 17 + 2 = 53$$
$$a(4) = 3 \cdot a(3) + 2 = 3 \cdot 53 + 2 = \boxed{161}$$

2. (8 points) Consider the following discrete dynamical system of two equations.

$$u(n) = u(n-1) + v(n-1) - 1$$
$$v(n) = u(n-1) + 2v(n-1)$$

If $u(0) = 2$ and $v(0) = 0$, then what are the values of $u(3)$ and $v(3)$?

$$u(1) = u(0) + v(0) - 1 = 2 + 0 - 1 = 1 \quad \text{and} \quad v(1) = u(0) + 2 \cdot v(0) = 2 + 2 \cdot 0 = 2$$
$$u(2) = u(1) + v(1) - 1 = 1 + 2 - 1 = 2 \quad \text{and} \quad v(2) = u(1) + 2 \cdot v(1) = 1 + 2 \cdot 2 = 5$$
$$u(3) = u(2) + v(2) - 1 = 2 + 5 - 1 = \boxed{6} \quad \text{and} \quad v(3) = u(2) + 2 \cdot v(2) = 2 + 2 \cdot 5 = \boxed{12}$$

3. (8 points) Suppose $p(n)$ represents some population n years from now, and that this population is modeled by the following discrete dynamical system.

$$p(n) = p(n-1) + 20$$
$$p(0) = 25$$

Which one of the following statements follows from this model?

The correct answer is: (b) The population will increase by 20 people per year.

4. (8 points) Suppose $a(n)$ represents the number of milligrams of some drug in the bloodstream n hours from now, and that the amount of this drug in the bloodstream is modeled by the following discrete dynamical system.

$$a(n) = 0.6a(n-1)$$
$$a(0) = 50$$

Which one of the following statements follows from this model?

The correct answer is: (d) The amount of drug in the bloodstream will decrease by 40% per hour.

5. (8 points) Find the equilibrium value in the following discrete dynamical system.

$$u(n) = 3u(n-1) - 100$$

Solving $E = 3E - 100$, we obtain the equilibrium value $\boxed{E = 50}$

6. (8 points) A discrete dynamical system for two functions $u(n)$ and $v(n)$ has precisely one equilibrium point. Some values for these functions are shown in the following tables with various initial values.

n	u(n)	v(n)
0	3.0	9.0
1	1.6	11.2
2	2.1	14.4
3	5.9	16.4
4	12.5	14.2
5	18.1	5.9
6	16.7	-6.4

n	u(n)	v(n)
0	4.5	10.0
1	4.4	10.5
2	4.8	11.0
3	5.7	11.0
4	6.9	10.1
5	7.4	8.2
6	6.0	6.2

n	u(n)	v(n)
0	5.0	10.0
1	5.0	10.0
2	5.0	10.0
3	5.0	10.0
4	5.0	10.0
5	5.0	10.0
6	5.0	10.0

n	u(n)	v(n)
0	5.2	12.0
1	7.2	11.4
2	9.1	8.9
3	8.8	5.0
4	4.6	2.2
5	-3.3	4.2
6	-10.7	13.7

(a) Determine the equilibrium point for these functions. Give your answer in the usual way as an ordered pair (E, F) where E goes with the function $u(n)$ and F goes with the function $v(n)$.

$(E, F) = (5, 10)$

(b) From the limited data in these tables, does the equilibrium point appear to be stable or unstable?

unstable

7. (8 points) A discrete dynamical system for $u(n)$ has an equilibrium value of 200. With an initial value of $u(0) = 250$, we have the following table of values for $u(n)$.

n	u(n)
0	250
1	220
2	208
3	203.2
4	201.28
5	200.512
6	200.2048

Determine as a percentage how much closer $u(n)$ is to equilibrium than $u(n-1)$ is to equilibrium.

From the limited data in this table it appears that

$$\lim_{n\to\infty} \frac{u(n)-20}{u(n-1)-20} = 0.4 \quad \text{so} \quad \boxed{60\% \text{ closer}}$$

8. (8 points) For most people, the half-life of caffeine in the bloodstream is approximately 4 hours. After drinking a cup of coffee with 100 mg of caffeine, how many hours will it take until only 10 mg of caffeine remain in the bloodstream? For full credit, your answer should be correctly rounded off to one place after the decimal point.

$$u(n) = 100a^n$$
$$50 = 100a^4$$
$$a = 0.5^{(1/4)} \approx 0.8409$$
$$u(n) \approx 100(0.8409)^n$$
$$10 \approx 100(0.8409)^n$$
$$n \approx \frac{\ln 0.1}{\ln 0.8409} \approx \boxed{13.3 \text{ hours}}$$

9. (5 points each) Find an explicit formula for each of the following discrete dynamical systems.

   (a) $u(n) = u(n-1) + 2 \quad \text{and} \quad u(0) = 3$
   $u(n) = 2n + 3$

   (b) $u(n) = 2u(n-1) \quad \text{and} \quad u(0) = 3$
   $u(n) = 3 \cdot 2^n$

   (c) $u(n) = 1.5u(n-1) - 10 \quad \text{and} \quad u(0) = 25$
   Solving $E = 1.5E - 10$, we obtain the equilibrium value $\boxed{E = 20}$.

   $$u(n) = c(a)^n + E$$
   $$u(n) = c(1.5)^n + 20$$

   Since $u(0) = 25$ we obtain $\boxed{u(n) = 5(1.5)^n + 20}$

10. (8 points) Let $u(n)$ represent the number of milligrams of some drug in the bloodstream n days after an initial dose of 200 mg. Suppose 25% of this drug is eliminated via the kidneys each day, but the remainder is reinforced with a maintenance dose of 120 mg per day.

    Find a discrete dynamical system along with an initial value for $u(n)$.

$$u(n) = 0.75u(n-1) + 120$$
$$u(0) = 200$$

11. (6 points) Suppose you borrow $\$25,000$ at a 7.2% annual interest rate compounded monthly to be paid back in monthly payments of \$800.

    (a) Write down a discrete dynamical system with initial condition to represent the balance owed on the loan after making n monthly payments.

    Noting that $0.072/12 = 0.006$ we obtain

$$u(n) = 1.006u(n-1) - 800$$
$$u(0) = 25000$$

    (b) How many months will it take to pay back the loan?

    35

    (c) The last payment will be a bit different than each of the preceding monthly payments. What will be the amount of this last payment?

    The first 34 monthly payments will be for \$800, but the 35th payment will be for \$568.71.

12. (7 points) Suppose $u(n)$ represents the amount of some drug in the bloodstream n hours after the initial dose. Find the best-fitting exponential function for $u(n)$ given the following set of data. Each number in your formula should be correctly rounded off to two places after the decimal point.

n	0	1	2	3
$u(n)$	200	125	86	60

$u(n) = 193.78(0.67)^n$