Math 172 **Test 2 (part 1 – no calculators) Solutions** **Spring 2006**

1. (7 points) On the graph of $y = 3x^2 - 100$, what is the slope of the curve at $x = 10$?

   Since $y' = 6x$ we obtain a slope of 60 at $x = 10$.

2. (7 points) If $y = 10e^{2x}$, then $\dfrac{dy}{dx} = 20e^{2x}$

3. (6 points) If $P(t) = t^3e^{-t}$, then $P'(t) = 3t^2e^{-t} - t^3e^{-t}$

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1. (8 points) Given the following initial value problem, use Euler's Method with $\Delta t = 2.5$ to make an estimate for $P(10)$.

$$\frac{dP}{dt} = \frac{1000}{P} \qquad \text{and} \qquad P(0) = 100$$

t_{old}	P_{old}	P'_{old}	$P_{new} \approx P_{old} + P'_{old} \cdot \Delta t$
0	100	10 $\left(\frac{1000}{100}\right)$	125 $(100 + 10 \cdot 2.5)$
2.5	125	8 $\left(\frac{1000}{125}\right)$	145 $(125 + 8 \cdot 2.5)$
5	145	6.897 $\left(\frac{1000}{145}\right)$	162.241 $(145 + 6.897 \cdot 2.5)$
7.5	162.241	6.164 $\left(\frac{1000}{162.241}\right)$	177.651 $(162.241 + 6.164 \cdot 2.5)$
10	177.651		

Using Euler's Method with $\Delta t = 2.5$, we obtain the estimate $P(10) \approx 177.651$

2. (6 points) It has been raining heavily since midnight. The function $f(t)$ represents the total number of inches of rain which have fallen in the t hours since midnight. Suppose that $f'(4) = 2$. Which of the following sentences must be true?

   The correct answer is (f) At 4:00AM it was raining at a rate of 2 inches per hour.

3. (9 points) Find all equilibrium values for the following differential equation. There is no need to discuss whether or not these equilibrium values are stable.

$$\frac{dP}{dt} = 2P(P-4)(3P-1)$$

   Solving $\dfrac{dP}{dt} = 0$ gives us the three equilibrium values $P = 0$, $P = 4$, and $P = 1/3$

4. (8 points) Suppose y is a function of t for which $\dfrac{dy}{dt} = 0.3(y-4)(y-8)$. Sketch rough graphs of y for each initial value below. Note: To save time you may draw all 4 together on one set of coordinate axes.

(a) $y(0) = 9$

(b) $y(0) = 8$

(c) $y(0) = 7$

(d) $y(0) = 0$

graph discussed in class

5. (9 points) Suppose that 50 rabbits are released on Lady Tottington's estate, and that the population of these rabbits grows according to an exponential model with a continuous growth rate of 6% per month.

   (a) Determine a differential equation with initial condition to model this population of rabbits.
   $\frac{dR}{dt} = 0.06R$ and $R(0) = 50$

   (b) Find an explicit formula for this population of rabbits.
   $R(t) = 50e^{0.06t}$

   (c) Sketch a rough graph for this population of rabbits, but be sure that your graph clearly shows the value of any intercepts and any long term behaviour.
   graph discussed in class

6. (12 points) The population of a town is currently 400, but is projected to grow by 20 people per year from now on.

   (a) Determine a differential equation with initial condition to model this town's population.
   $\frac{dP}{dt} = 20$ and $P(0) = 400$

   (b) Determine a discrete dynamical system with initial condition to model this town's population.
   $P(n) = P(n-1) + 20$ and $P(0) = 400$

   (c) Find an explicit formula for this town's population.
   $P(n) = 20n + 400$

   (d) Sketch a rough graph for this town's population, but be sure that your graph clearly shows the value of any intercepts and any long term behaviour.
   graph discussed in class

7. (6 points) Suppose that 100 wild pigs are released on an island, and that the population of these pigs grows according to a logistic model with an intrinsic growth rate of 4% per year and a carrying capacity of 800.

   (a) Determine a differential equation with initial condition to model this population of wild pigs.
   $\frac{dP}{dt} = 0.04P\left(1 - \frac{P}{800}\right)$ and $P(0) = 100$

(b) Sketch a rough graph for this population of pigs, but be sure that your graph clearly shows the value of any intercepts and any long term behaviour.

graph discussed in class

8. (6 points) The heart mass of a mammal is proportional to its body mass. A human with a body mass of 70 kilograms has a heart mass of 0.42 kilograms.

(a) Write a formula for the heart mass, H, of a mammal as a function of its body mass, B.

$$H = k \cdot B$$

$$0.42 = k \cdot 70 \Rightarrow k = 0.006$$

Thus $\boxed{\text{H=0.006B}}$

(b) Estimate the heart mass of a horse with a body mass of 650 kilograms.

$$H = 0.006(650) = 3.9 \text{ kg}$$

9. (6 points) Suppose a certain chemical is eliminated from the body by the kidneys and the liver. Let $u(n)$ represent the amount of this chemical in a person's bloodstream after n days. Assume that each day, the kidneys remove 25% of the chemical from the blood. Also assume that each day, the fraction of the chemical that is broken down by enzymes from the liver is given by

$$\frac{3}{7 + u(n-1)}$$

Finally, assume that each day, the person takes a dose of 200 mg of this chemical. Develop a dynamical system for $u(n)$. You do not need an initial value.

$$u(n) = u(n-1) - 0.25u(n-1) - \frac{3}{7 + u(n-1)} \cdot u(n-1) + 200$$

$$u(n) = 0.75u(n-1) - \frac{3u(n-1)}{7 + u(n-1)} + 200$$

10. (10 points) For Mary's metabolism, the dynamical system modeling her elimination of alcohol is

$$a(n) = a(n-1) - \frac{9a(n-1)}{4.2 + a(n-1)} + d$$

where $a(n)$ is the amount of alcohol (in grams) in her bloodstream after n hours of drinking d grams of alcohol per hour. For Mary's weight, 38 grams of alcohol in the bloodstream represents a blood alcohol level of 0.08.

(a) Correct to one place after the decimal point, what is the largest number of grams of alcohol Mary can drink per hour at a 3-hour party if she wants to stay below that blood alcohol level of 0.08? Begin with $a(0) = 0$.

This problem can be solved by trying different values for d, entering the dynamical system on your calculator, and looking at the table of values.

The following tables show that Mary can consume somewhere between 17 and 18 grams of alcohol per hour, but still have less than 38 grams of alcohol in her bloodstream after 3 hours.

# hours	alcohol in blood ($d = 17$)
0	0
1	17
2	26.783
3	36.003

# hours	alcohol in blood ($d = 18$)
0	0
1	18
2	28.703
3	38.852

The following tables show that Mary can consume somewhere between 17.7 and 17.8 grams of alcohol per hour, but still have less than 38 grams of alcohol in her bloodstream after 3 hours.

# hours	alcohol in blood ($d = 17.7$)
0	0
1	17.7
2	28.126
3	37.995

# hours	alcohol in blood ($d = 17.8$)
0	0
1	17.8
2	28.318
3	38.281

So to a reasonable degree of accuracy, Mary can drink at most 17.7 grams of alcohol each hour.

(b) Using that a 12-ounce can of beer contains about 14 grams of alcohol, convert your answer in part (a) to ounces of beer.

$$17.7 \text{ grams} \left(\frac{12 \text{ ounces}}{14 \text{ grams}} \right) \approx 15.17 \text{ ounces}$$