Math 172 (Section 1) Test 1 Spring 2009

Name SOLUTIONS — WHITE VERSION

Seat # ______________________

- Do not open this test booklet until told to do so.
- Turn off all cell phones.
- For multiple-choice questions, precisely one answer is correct. Circle this correct answer.
- For all other questions, you must show sufficient work to justify your answer.
- You are not allowed to borrow another student's calculator during the test.
- Show your Student ID when you turn in your test.

Do not write below this line

#1 (5 points) ______________

#2 (15 points) ______________

#3 (8 points) ______________

#4 (15 points) ______________

#5 (8 points) ______________

#6 (8 points) ______________

#7 (5 points) ______________

#8 (5 points) ______________

#9 (10 points) ______________

#10 (15 points) ______________

#11 (6 points) ______________

Total (100 points) ______________

1. (5 points) Consider the following discrete dynamical system.

$$P(t+1) = 1.03P(t) + 60$$

If $P(0) = 400$, then what is the value of $P(9)$?

t	$P(t)$
0	400
1	472
⋮	⋮
9	1131.5

$P(9) \approx 1131.5$

2. (15 points) Suppose that a patient takes a daily dose of 12 milligrams of some drug, and that each day the kidneys filter out 20% of this drug from the patient's bloodstream.

(a) Determine a discrete dynamical system along with an initial value for $d(t)$, the number of milligrams of this drug in this patient's bloodstream t days after the initial dose.

$d(t) = d(t-1) - 0.2\, d(t-1) + 12$

$d(t) = 0.8\, d(t-1) + 12$

$d(0) = 12$

(b) How many milligrams of this drug are in the bloodstream 4 days after the initial dose?

t	$d(t)$
0	12
1	21.6
2	29.28
3	35.424
4	40.339

$d(4) \approx 40.3$ mg

(c) Assuming that the patient is to continue taking this drug for a long time, what was the doctor's **target goal** for the desired amount of this drug in the bloodstream?

t	$d(t)$
0	12
10	54.846
20	59.447
30	59.941
40	59.994
50	59.999
⋮	↓
	60

$\lim_{t \to \infty} d(t) = 60$ mg

or solve for the stable equilibrium value $d^* = 60$ mg

3. (8 points) There are currently 600 people living in a small town which is projected to grow by 5% each year. Let $P(t)$ represent the population of the town t years from now.

(a) Find a discrete dynamical system along with an initial value for $P(t)$.

$$P(t) = P(t-1) + 0.05P(t-1)$$

$$P(t) = 1.05P(t-1)$$

$$P(0) = 600$$

(b) In how many years from now will the population reach 2000 people?

t	$P(t)$
0	600
1	630
⋮	⋮
24	1935.1
25	2031.8

Between 24 and 25 years

or more precisely,

$$P(t) = 600(1.05)^t$$

$$2000 = 600(1.05)^t$$

$$3.\overline{3} = 1.05^t$$

$$\ln(3.\overline{3}) = t\ln(1.05)$$

$$t = \frac{\ln(3.\overline{3})}{\ln(1.05)} \approx 24.7 \text{ years}$$

4. (15 points) Each dynamical system below has an equilibrium value of 100. Determine if this equilibrium value is stable or unstable for each system. You do not need to show your work here.

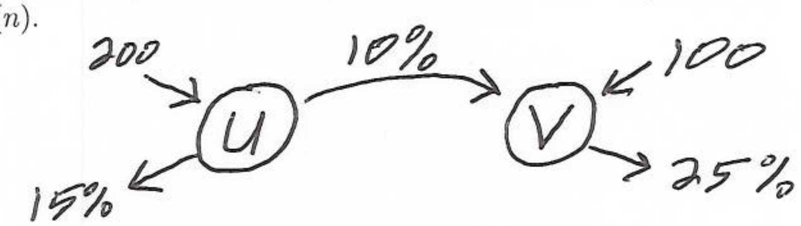
(a) $u(n) = -0.8u(n-1) + 180$ stable since $|-0.8| < 1$

(b) $P(t) = 1.04P(t-1) - 4$ unstable since $|1.04| > 1$

(c) $v(n+1) = 0.93v(n) + 7$ stable since $|0.93| < 1$

Alternate approach: using a starting value not equal to 100, make a table of values to see which functions approach 100.

5. (8 points) There are 2 drugs, U and V. Let $u(n)$ and $v(n)$ represent the number of milligrams of each drug in the body at the beginning of day n. The liver converts 10% of U into V each day. The kidneys remove 15% of U and 25% of V each day. Assume that 200 mg of U and 100 mg of V are consumed each day. Develop a discrete dynamical system (without initial values) to represent $u(n)$ and $v(n)$.



$$u(n) = u(n-1) - 0.1u(n-1) - 0.15u(n-1) + 200$$
$$v(n) = v(n-1) - 0.25v(n-1) + 0.1u(n-1) + 100$$

$$u(n) = 0.75u(n-1) + 200$$
$$v(n) = 0.1u(n-1) + 0.75v(n-1) + 100$$

6. (8 points) Suppose that $u(n)$ and $v(n)$ represent the number of milligrams of drugs U and V, respectively, in the bloodstream n days after an initial dose of each. The discrete dynamical system for this is shown below.

$$u(n) = 0.7u(n-1) + 0.05v(n-1) + 19.5$$
$$v(n) = 0.1u(n-1) + 0.8v(n-1) + 10$$
$$u(0) = 40$$
$$v(0) = 20$$

(a) How many milligrams of each drug are in the bloodstream 5 days after the initial dose?

$u(5) \approx 66.8$ mg $v(5) \approx 59.0$ mg

(b) Assuming that the patient is to continue taking this drug for a long time, what was the doctor's **target goal** for the desired amount of each drug in the bloodstream?

$\lim_{n \to \infty} (u(n), v(n)) = (80, 90)$ from table

or finding stable equilibrium is $(u^*, v^*) = (80, 90)$

gives 80 mg of drug U and 90 mg of drug V

7. (5 points) Suppose $P(t)$ represents some population t years from now, and that this population is modeled by the following discrete dynamical system.

$$P(t) = P(t-1) + 60$$
$$P(0) = 30$$

Which one of the following statements follows from this model?

(a) The population will increase by 30% per year.

(b) The population will increase by 30 people per year.

(c) The population will increase by 40% per year.

(d) The population will increase by 40 people per year.

(e) The population will increase by 60% per year.

(f) The population will increase by 60 people per year.

(g) The population will increase by 70% per year.

(h) The population will increase by 70 people per year.

8. (5 points) Suppose $a(n)$ represents the number of milligrams of some drug in the bloodstream n hours from now, and that the amount of this drug in the bloodstream is modeled by the following discrete dynamical system.

$$a(n) = 0.25a(n-1)$$
$$a(0) = 50$$

Which one of the following statements follows from this model?

(a) The amount of drug in the bloodstream will increase by 25% per hour.

(b) The amount of drug in the bloodstream will decrease by 25% per hour.

(c) The amount of drug in the bloodstream will increase by 25 milligrams per hour.

(d) The amount of drug in the bloodstream will decrease by 25 milligrams per hour.

(e) The amount of drug in the bloodstream will increase by 50% per hour.

(f) The amount of drug in the bloodstream will decrease by 50% per hour.

(g) The amount of drug in the bloodstream will increase by 50 milligrams per hour.

(h) The amount of drug in the bloodstream will decrease by 50 milligrams per hour.

(i) The amount of drug in the bloodstream will increase by 75% per hour.

(j) The amount of drug in the bloodstream will decrease by 75% per hour.

(k) The amount of drug in the bloodstream will increase by 75 milligrams per hour.

(l) The amount of drug in the bloodstream will decrease by 75 milligrams per hour.

9. (10 points) Compute the equilibrium point (p^*, q^*) for the following discrete dynamical system.

$$p(t+1) = 1.5p(t) - 2q(t)$$
$$q(t+1) = 0.5p(t) - 1.2q(t) + 1.5$$

$p^* = 1.5p^* - 2q^*$
$q^* = 0.5p^* - 1.2q^* + 1.5$

$2q^* = 1.5p^* - p^*$
$2q^* = 0.5p^*$
$4q^* = p^*$

$q^* = 0.5(4q^*) - 1.2q^* + 1.5$
$q^* = 0.8q^* + 1.5$
$0.2q^* = 1.5$
$q^* = 7.5$

$p^* = 4(7.5) = 30$

$(p^*, q^*) = (30, 7.5)$

Check table of values with $p(0) = 30$
and $q(0) = 7.5$

t	$p(t)$	$q(t)$
0	30	7.5
1	30	7.5
2	30	7.5
⋮	⋮	⋮

✓

10. (15 points) Find an explicit formula for an expression which satisfies each of the following discrete dynamical systems.

(a) $Q(t) = Q(t-1) - 8$ and $Q(0) = 120$

$Q(t) = -8t + 120$

(b) $b(n) = 1.08b(n-1)$ and $b(0) = 120$

$b(n) = 120(1.08)^n$

(c) $h(s) = 1.2h(s-1) - 8$ and $h(0) = 10$

$h^* = 1.2h^* - 8$

$8 = 0.2h^*$

$h^* = 40$

$h(s) = c(1.2)^s + 40$

$10 = c(1.2)^0 + 40 \Rightarrow c = -30$

$h(s) = -30(1.2)^s + 40$

11. (6 points) We are given the following discrete dynamical system.

$$u(n) = 0.7u(n-1) + 0.025v(n-1) + 50$$
$$v(n) = 1.2u(n-1) + 0.9v(n-1) + 80$$

Although the function $u(n)$ is not linear, as n increases the function starts to look like a line with slope 17.5.

choosing arbitrary starting values such as $u(0)=10$ and $v(0)=20$ we obtain:

$$u(1)-u(0) = 47.5$$
$$u(2)-u(1) = 35.5$$
$$\vdots$$
$$u(10)-u(9) \approx 17.80233088$$
$$\vdots$$
$$u(20)-u(19) \approx 17.50182808$$
$$\vdots$$
$$u(30)-u(29) \approx 17.50001105$$
$$\vdots$$
$$u(40)-u(39) \approx 17.50000007$$
$$\vdots \qquad \vdots$$

$$\lim_{n\to\infty} \left(u(n)-u(n-1) \right) = 17.5$$