

Name SOLUTIONS - GREEN VERSION

Seat # _____

- Do not open this test booklet until told to do so.
- Turn off all cell phones.
- For multiple-choice questions, precisely one answer is correct. Circle this correct answer.
- For all other questions, you must show sufficient work to justify your answer.
- You are not allowed to borrow another student's calculator during the test.
- Show your Student ID when you turn in your test.

Do not write below this line

#1 (5 points) _____

#7 (5 points) _____

#2 (15 points) _____

#8 (5 points) _____

#3 (8 points) _____

#9 (10 points) _____

#4 (15 points) _____

#10 (15 points) _____

#5 (8 points) _____

#11 (6 points) _____

#6 (8 points) _____

Total (100 points) _____

1. (5 points) Consider the following discrete dynamical system.

$$P(t+1) = 1.06P(t) + 50$$

If $P(0) = 300$, then what is the value of $P(9)$?

t	$P(t)$
0	300
1	368
\vdots	\vdots
9	1081.4

$$P(9) \approx 1081.4$$

2. (15 points) Suppose that a patient takes a daily dose of 16 milligrams of some drug, and that each day the kidneys filter out 25% of this drug from the patient's bloodstream.

- (a) Determine a discrete dynamical system along with an initial value for $d(t)$, the number of milligrams of this drug in this patient's bloodstream t days after the initial dose.

$$d(t) = d(t-1) - 0.25d(t-1) + 16$$

$$d(t) = 0.75d(t-1) + 16$$

$$d(0) = 16$$

- (b) How many milligrams of this drug are in the bloodstream 4 days after the initial dose?

t	$d(t)$
0	16
1	28
2	37
3	43.75
4	48.8125

$$d(4) \approx 48.8 \text{ mg}$$

- (c) Assuming that the patient is to continue taking this drug for a long time, what was the doctor's target goal for the desired amount of this drug in the bloodstream?

t	$d(t)$
0	16
7	57.593
14	63.145
21	63.886
28	63.985
35	63.998
\vdots	\downarrow
	64

$$\lim_{t \rightarrow \infty} d(t) = 64 \text{ mg}$$

or solve for the stable equilibrium value $d^* = 64 \text{ mg}$

3. (8 points) There are currently 800 people living in a small town which is projected to grow by 4% each year. Let $P(t)$ represent the population of the town t years from now.

(a) Find a discrete dynamical system along with an initial value for $P(t)$.

$$P(t) = P(t-1) + 0.04P(t-1)$$

$$P(t) = 1.04P(t-1)$$

$$P(0) = 800$$

(b) In how many years from now will the population reach 2000 people?

t	$P(t)$
0	800
1	832
\vdots	
23	1971.8
24	2050.6

Between 23 and 24 years

or more precisely,

$$P(t) = 800(1.04)^t$$

$$2000 = 800(1.04)^t$$

$$2.5 = 1.04^t$$

$$\ln(2.5) = t \cdot \ln(1.04)$$

$$t = \frac{\ln(2.5)}{\ln(1.04)} \approx 23.4 \text{ years}$$

4. (15 points) Each dynamical system below has an equilibrium value of 100. Determine if this equilibrium value is stable or unstable for each system. You do not need to show your work here.

(a) $u(n) = -0.8u(n-1) + 180$

stable since $|-0.8| < 1$

(b) $P(t) = 1.04P(t-1) - 4$

unstable since $|1.04| > 1$

(c) $v(n+1) = 0.93v(n) + 7$

stable since $|0.93| < 1$

Alternate approach: using a starting value not equal to 100, make a table of values to see which functions approach 100.

5. (8 points) There are 2 drugs, U and V . Let $u(n)$ and $v(n)$ represent the number of milligrams of each drug in the body at the beginning of day n . The liver converts 15% of U into V each day. The kidneys remove 10% of U and 20% of V each day. Assume that 300 mg of U and 200 mg of V are consumed each day. Develop a discrete dynamical system (without initial values) to represent $u(n)$ and $v(n)$.



$$u(n) = u(n-1) - 0.1u(n-1) - 0.15u(n-1) + 300$$

$$v(n) = v(n-1) - 0.2v(n-1) + 0.15u(n-1) + 200$$

$$u(n) = 0.75u(n-1) + 300$$

$$v(n) = 0.15u(n-1) + 0.8v(n-1) + 200$$

6. (8 points) Suppose that $u(n)$ and $v(n)$ represent the number of milligrams of drugs U and V , respectively, in the bloodstream n days after an initial dose of each. The discrete dynamical system for this is shown below.

$$u(n) = 0.7u(n-1) + 0.05v(n-1) + 19.5$$

$$v(n) = 0.1u(n-1) + 0.8v(n-1) + 10$$

$$u(0) = 50$$

$$v(0) = 30$$

- (a) How many milligrams of each drug are in the bloodstream 5 days after the initial dose?

$$u(5) \approx 69.5 \text{ mg} \quad v(5) \approx 64.1 \text{ mg}$$

- (b) Assuming that the patient is to continue taking this drug for a long time, what was the doctor's target goal for the desired amount of each drug in the bloodstream?

$$\lim_{n \rightarrow \infty} (u(n), v(n)) = (80, 90) \text{ from table}$$

or finding stable equilibrium value $(u^*, v^*) = (80, 90)$

gives 80 mg of drug U and 90 mg of drug V

7. (5 points) Suppose $P(t)$ represents some population t years from now, and that this population is modeled by the following discrete dynamical system.

$$P(t) = P(t - 1) + 70$$

$$P(0) = 40$$

Which one of the following statements follows from this model?

- (a) The population will increase by 30% per year.
 - (b) The population will increase by 30 people per year.
 - (c) The population will increase by 40% per year.
 - (d) The population will increase by 40 people per year.
 - (e) The population will increase by 60% per year.
 - (f) The population will increase by 60 people per year.
 - (g) The population will increase by 70% per year.
 - (h) The population will increase by 70 people per year.
8. (5 points) Suppose $a(n)$ represents the number of milligrams of some drug in the bloodstream n hours from now, and that the amount of this drug in the bloodstream is modeled by the following discrete dynamical system.

$$a(n) = 0.25a(n - 1)$$

$$a(0) = 50$$

Which one of the following statements follows from this model?

- (a) The amount of drug in the bloodstream will increase by 25% per hour.
- (b) The amount of drug in the bloodstream will decrease by 25% per hour.
- (c) The amount of drug in the bloodstream will increase by 25 milligrams per hour.
- (d) The amount of drug in the bloodstream will decrease by 25 milligrams per hour.
- (e) The amount of drug in the bloodstream will increase by 50% per hour.
- (f) The amount of drug in the bloodstream will decrease by 50% per hour.
- (g) The amount of drug in the bloodstream will increase by 50 milligrams per hour.
- (h) The amount of drug in the bloodstream will decrease by 50 milligrams per hour.
- (i) The amount of drug in the bloodstream will increase by 75% per hour.
- (j) The amount of drug in the bloodstream will decrease by 75% per hour.
- (k) The amount of drug in the bloodstream will increase by 75 milligrams per hour.
- (l) The amount of drug in the bloodstream will decrease by 75 milligrams per hour.

9. (10 points) Compute the equilibrium point (p^*, q^*) for the following discrete dynamical system.

$$\begin{aligned} p(t+1) &= 1.5p(t) - 2q(t) \\ q(t+1) &= 0.5p(t) - 1.2q(t) + 1.5 \end{aligned}$$

$$p^* = 1.5p^* - 2q^*$$

$$q^* = 0.5p^* - 1.2q^* + 1.5$$

$$\rightarrow 2q^* = 1.5p^* - p^*$$

$$2q^* = 0.5p^*$$

$$p^* = 4q^*$$

$$q^* = 0.5(4q^*) - 1.2q^* + 1.5 \leftarrow$$

$$q^* = 0.8q^* + 1.5$$

$$0.2q^* = 1.5$$

$$q^* = 7.5$$

$$\rightarrow p^* = 4(7.5) = 30$$

$$(p^*, q^*) = (30, 7.5)$$

check table of values with $p(0) = 30$
and $q(0) = 7.5$

t	$p(t)$	$q(t)$
0	30	7.5
1	30	7.5
2	30	7.5
\vdots	\vdots	\vdots
\vdots	\vdots	\vdots



10. (15 points) Find an explicit formula for an expression which satisfies each of the following discrete dynamical systems.

(a) $Q(t) = Q(t-1) - 4$ and $Q(0) = 160$

$$Q(t) = -4t + 160$$

(b) $b(n) = 1.07b(n-1)$ and $b(0) = 140$

$$b(n) = 140(1.07)^n$$

(c) $h(s) = 1.2h(s-1) - 4$ and $h(0) = 5$

$$h^* = 1.2h^* - 4$$

$$4 = 0.2h^*$$

$$h^* = 20$$

$$h(s) = C(1.2)^s + 20$$

$$5 = C(1.2)^0 + 20 \Rightarrow C = -15$$

$$h(s) = -15(1.2)^s + 20$$

11. (6 points) We are given the following discrete dynamical system.

$$u(n) = 0.7u(n-1) + 0.025v(n-1) + 50$$

$$v(n) = 1.2u(n-1) + 0.9v(n-1) + 80$$

Although the function $v(n)$ is not linear, as n increases the function starts to look like a line with slope 210.

Choosing arbitrary starting values such as $u(0) = 5$ and $v(0) = 10$ we obtain

$$v(1) - v(0) = 85$$

$$v(2) - v(1) = 135$$

$$\vdots$$
$$v(10) - v(9) \approx 208.740288$$

$$\vdots$$
$$v(20) - v(19) \approx 209.992383$$

$$\vdots$$
$$v(30) - v(29) \approx 209.9999539$$

$$\vdots$$
$$v(40) - v(39) \approx 209.9999997$$

\vdots

\vdots

$$\lim_{n \rightarrow \infty} (v(n) - v(n-1)) = 210$$