Math 172 **Worksheet D Solutions** Spring 2008

1. For each discrete dynamical system do the following.

   - Quickly decide if the system represents a linear, exponential, or shifted exponential function.
   - Quickly decide if the system has a stable equilibrium value, an unstable equilibrium value, or no equilibrium value.
   - Find an explicit formula for the system.
   - For each system that has an equilibrium value, determine as a percentage how much closer to or further away from the equilibrium value the function moves each time period.

   (a) $P(t+1) = P(t) - 20$ and $P(0) = 350$

- LINEAR
- NO EQUILIBRIUM (SINCE SLOPE NONZERO)
- $P(t) = -20t + 350$
-

(b) $v(n+1) = \dfrac{v(n)}{4}$ and $v(0) = 1600$

$v(n+1) = 0.25\, v(n)$

- EXPONENTIAL
- $V^* = 0$ IS A STABLE EQUILIBRIUM VALUE SINCE $-1 < 0.25 < 1$.
- $v(n) = 1600\,(0.25)^n$
- $\displaystyle \lim_{n\to\infty} \frac{v(n) - 0}{v(n-1) - 0} = \lim_{n\to\infty} \frac{0.25\,v(n-1)}{v(n-1)} = 0.25$

SO THE FUNCTION V MOVES 75% CLOSER TO EQUILIBRIUM EACH TIME PERIOD.

(c) $q(t) = q(t-1) + 2.5$ and $q(0) = 40$

- LINEAR
- NO EQUILIBRIUM (SINCE SLOPE NONZERO)
- $q(t) = 2.5t + 40$
-

(d) $h(s) = \frac{3}{2}h(s-1)$ and $h(0) = 45$

$h(s) = 1.5h(s-1)$

- EXPONENTIAL
- $h^* = 0$ IS AN UNSTABLE ~~[illegible]~~ EQUILIBRIUM VALUE SINCE $1.5 > 1$
- $h(s) = 45(1.5)^s$
- $\lim_{s\to\infty} \frac{h(s)-0}{h(s-1)-0} = \lim_{s\to\infty} \frac{1.5h(s-1)}{h(s-1)} = 1.5$

SO THE FUNCTION h MOVES 50% FURTHER AWAY FROM EQUILIBRIUM EACH TIME PERIOD.

(e) $Q(t+1) = 1.25Q(t) - 22.5$ and $Q(0) = 150$

- SHIFTED EXPONENTIAL
- $Q^* = 1.25Q^* - 22.5 \Rightarrow Q^* = 90$ WHICH IS AN UNSTABLE EQUIL. VALUE SINCE $1.25 > 1$
- $Q(t) = K(1.25)^t + 90$

$Q(0) = 150 \Rightarrow 150 = K(1.25)^0 + 90 \Rightarrow K = 60$

SO $Q(t) = 60(1.25)^t + 90$

- $\lim_{t\to\infty} \frac{Q(t)-90}{Q(t-1)-90} = \lim_{t\to\infty} \frac{1.25Q(t-1)-22.5-90}{Q(t-1)-90}$

$= \lim_{t\to\infty} \frac{1.25(Q(t-1)-90)}{Q(t-1)-90} = 1.25$

Q MOVES 25% FURTHER AWAY FROM EQUIL. EACH TIME PERIOD.

(f) $B(n) = 0.9B(n-1) + 30$ and $B(0) = 750$

- SHIFTED EXPONENTIAL
- $B^* = 0.9B^* + 30 \Rightarrow B^* = 300$ WHICH IS A STABLE EQUILIBRIUM VALUE SINCE $-1 < 0.9 < 1$
- $B(n) = K(0.9)^n + 300$

  $B(0) = 750 \Rightarrow 750 = K(0.9)^0 + 300 \Rightarrow K = 450$

  SO $B(n) = 450(0.9)^n + 300$

- $$\lim_{n\to\infty} \frac{B(n)-300}{B(n-1)-300} = \lim_{n\to\infty} \frac{0.9B(n-1)+30-300}{B(n-1)-300}$$
  $$= \lim_{n\to\infty} \frac{0.9(B(n-1)-300)}{B(n-1)-300} = 0.9$$

B MOVES 10% CLOSER TO EQUIL. EACH TIME PERIOD

(g) $d(n) = \cancel{1.3d(n) - 63}$ and $d(0) = 250$

$d(n) = 1.3d(n-1) - 63$

- SHIFTED EXPONENTIAL
- $d^* = 1.3d^* - 63 \Rightarrow d^* = 210$ WHICH IS AN UNSTABLE EQUIL. VALUE SINCE $1.3 > 1$
- $d(n) = K(1.3)^n + 210$

  $d(0) = 250 \Rightarrow 250 = K(1.3)^0 + 210 \Rightarrow K = 40$

  SO $d(n) = 40(1.3)^n + 210$

- $$\lim \frac{d(n)-210}{d(n-1)-210} = \lim \frac{1.3d(n-1)-63-210}{d(n-1)-210}$$
  $$= \lim \frac{1.3(d(n-1)-210)}{d(n-1)-210} = 1.3$$

d MOVES 30% FURTHER AWAY FROM EQUIL. EACH TIME PERIOD

(h) $w(t) = 0.85w(t-1) + 90$ and $w(0) = 200$

- SHIFTED EXPONENTIAL
- $W^* = 0.85W^* + 90 \Rightarrow W^* = 600$ WHICH IS A STABLE EQUILIBRIUM VALUE SINCE $-1 < 0.85 < 1$
- $w(t) = K(0.85)^t + 600$

  $w(0) = 200 \Rightarrow 200 = K(0.85)^0 + 600 \Rightarrow K = -400$

  SO $w(t) = -400(0.85)^t + 600$
- $$\lim_{t\to\infty} \frac{w(t)-600}{w(t-1)-600} = \lim_{t\to\infty} \frac{0.85w(t-1)+90-600}{w(t-1)-600}$$
  $$= \lim_{t\to\infty} \frac{0.85(w(t-1)-600)}{w(t-1)-600} = 0.85$$

W MOVES 15% CLOSER TO EQUIL. EACH TIME PERIOD.

(i) $p(n) = \dfrac{p(n-1) - 2500}{5}$ and $p(0) = 50$

- $P(n) = 0.2P(n-1) - 500$ SHIFTED EXPONENTIAL
- $P^* = 0.2P^* - 500 \Rightarrow P^* = -625$ WHICH IS A STABLE EQUILIBRIUM VALUE SINCE $-1 < 0.2 < 1$
- $P(n) = K(0.2)^n - 625$

  $P(0) = 50 \Rightarrow 50 = K(0.2)^0 - 625 \Rightarrow K = 675$

  SO $p(n) = 675(0.2)^n - 625$
- $$\lim_{n\to\infty} \frac{P(n)-(-625)}{P(n-1)-(-625)} = \lim_{n\to\infty} \frac{0.2p(n-1)-500+625}{P(n-1)+625}$$
  $$= \lim_{n\to\infty} \frac{0.2(p(n-1)+625)}{P(n-1)+625} = 0.2$$

P MOVES 80% CLOSER TO EQUIL. EACH TIME PERIOD

2. The dynamical system shown has a stable equilibrium point at $(p^*, q^*) = (280, 300)$. Given that $p(0) = 50$ and $q(0) = 23$, determine the eventual rate at which q approaches equilibrium. Show all calculations you made to find the rate.

$$p(t) = 0.5p(t-1) + 0.3q(t-1) + 50$$
$$q(t) = 0.4p(t-1) + 0.55q(t-1) + 23$$

$$\frac{q(1)-300}{q(0)-300} \approx 0.88213$$

$$\frac{q(2)-300}{q(1)-300} \approx 0.87499$$

$$\vdots \qquad \vdots$$

$$\frac{q(10)-300}{q(9)-300} \approx 0.87231$$

$$\vdots \qquad \vdots$$

$$\frac{q(20)-300}{q(19)-300} \approx 0.87231$$

$$\vdots \qquad \vdots$$

$$\frac{q(30)-300}{q(29)-300} \approx 0.87231$$

$$\vdots \qquad \vdots$$

IT APPEARS THAT

$$\lim_{t\to\infty} \frac{q(t)-300}{q(t-1)-300} \approx 0.872$$

SINCE $1 - 0.872 = 0.128$,

q WILL EVENTUALLY MOVE ABOUT 12.8% CLOSER TO EQUILIBRIUM EACH TIME PERIOD

3. The dynamical system shown has a stable equilibrium point at $(u^*, v^*) = (20, 32)$. Given that $u(0) = 10$ and $v(0) = 20$, determine the eventual rate at which u approaches equilibrium and the rate at which v approaches equilibrium. Show all calculations you made to find the rate.

$$u(n) = 0.3u(n-1) - 0.5v(n-1) + 30$$
$$v(n) = 0.2u(n-1) + v(n-1) - 4$$

$$\frac{u(1)-20}{u(0)-20} \approx -0.3$$

$$\frac{u(11)-20}{u(10)-20} \approx 0.803799$$

$$\frac{u(21)-20}{u(20)-20} \approx 0.800034$$

$$\frac{u(31)-20}{u(30)-20} \approx 0.800000$$

IT APPEARS THAT

$$\lim_{n \to \infty} \frac{u(n)-20}{u(n-1)-20} = 0.8$$

SO u WILL EVENTUALLY MOVE ABOUT 20% CLOSER TO EQUILIBRIUM EACH TIME PERIOD

$$\frac{v(1)-32}{v(0)-32} \approx 1.166$$

$$\frac{v(11)-32}{v(10)-32} \approx 0.801508$$

$$\frac{v(21)-32}{v(20)-32} \approx 0.800014$$

$$\frac{v(31)-32}{v(30)-32} \approx 0.800000$$

IT APPEARS THAT

$$\lim_{n \to \infty} \frac{v(n)-32}{v(n-1)-32} = 0.8$$

SO v WILL EVENTUALLY MOVE ABOUT 20% CLOSER TO EQUILIBRIUM EACH TIME PERIOD

4. For the following dynamical system, there is no equilibrium point, but the values for $u(n)$ eventually change by approximately the same amount.

$$u(n) = 0.9u(n-1) + 0.2v(n-1) + 600$$
$$v(n) = 0.1u(n-1) + 0.8v(n-1) + 400$$

(a) What is that approximate amount by which $u(n)$ eventually changes?

$u(1) - u(0) \approx 600$
$u(2) - u(1) \approx 620$
$\vdots$
$u(10) - u(9) \approx 663.976$
$\vdots$
$u(20) - u(19) \approx 666.591$
$\vdots$
$u(30) - u(29) \approx 666.665$
$\vdots$
$u(40) - u(39) \approx 666.6\cancel{6}606$
$\vdots \qquad \vdots$

SINCE NO INITIAL VALUES WERE GIVEN, I ARBITRARILY CHOSE $u(0) = 0$ AND $v(0) = 0$.

ANY OTHER STARTING VALUES WILL RESULT IN THE SAME LIMIT BELOW.

IT APPEARS THAT $\lim_{n\to\infty} (u(n) - u(n-1)) = 666.\overline{6}$

SO FOR LARGE VALUES OF n,
$u(n) \approx u(n-1) + 666.\overline{6}$

(b) Does $u(n)$ appear to be more linear or exponential for large n?

$u(n)$ APPEARS LINEAR WITH SLOPE EQUAL TO $666.\overline{6}$ FOR LARGE n

5. For the following dynamical system, there is no equilibrium point, but the values for $p(t)$ eventually change by approximately the same amount.

$$p(t) = 1.2p(t-1) + 0.6q(t-1) + 10$$
$$q(t) = -0.2p(t-1) + 0.4q(t-1) + 20$$

(a) What is that approximate amount by which $p(t)$ eventually changes?

ARBITRARILY LETTING $p(0)=0$ AND $q(0)=0$, WE FIND:

$p(1)-p(0) \approx 10$

$p(11)-p(10) \approx 44.78837$

$p(21)-p(20) \approx 44.99872$

$p(31)-p(30) \approx 44.99999$

IT APPEARS THAT $\lim_{t\to\infty}(p(t)-p(t-1)) = 45$

SO $p(t)$ EVENTUALLY CHANGES BY 45 EACH TIME PERIOD.

(b) Does $p(t)$ appear to be more linear or exponential for large t?

$p(t)$ APPEARS LINEAR WITH SLOPE 45 FOR LARGE t.

(c) Answer questions (a) and (b) for the function $q(t)$.

$q(1)-q(0) \approx 20$

$q(11)-q(10) \approx -14.78837$

$q(21)-q(20) \approx -14.99872$

$q(31)-q(30) \approx -14.99999$

IT APPEARS THAT $\lim_{t\to\infty}(q(t)-q(t-1)) = -15$

SO $q(t)$ EVENTUALLY APPEARS LINEAR WITH SLOPE -15