

1. Suppose that  $y$  is a function of  $x$  which satisfies the differential equation

$$\frac{dy}{dx} = 3(x - 2)(x - 4)$$

- (a) Specify an interval of either  $x$ -values or  $y$ -values for which the graph of  $y$  will be increasing.

$$x < 2 \text{ or } x > 4$$

- (b) Specify an interval of either  $x$ -values or  $y$ -values for which the graph of  $y$  will be decreasing.

$$2 < x < 4$$

2. Suppose that  $y$  is a function of  $x$  which satisfies the differential equation

$$\frac{dy}{dx} = 3(y - 2)(y - 4)$$

- (a) Specify an interval of either  $x$ -values or  $y$ -values for which the graph of  $y$  will be increasing.

$$y < 2 \text{ or } y > 4$$

- (b) Specify an interval of either  $x$ -values or  $y$ -values for which the graph of  $y$  will be decreasing.

$$2 < y < 4$$

3. Consider the population model for logistic growth

$$\frac{dP}{dt} = 0.15P \left( 1 - \frac{P}{6000} \right)$$

where  $P$  is the population at time  $t$ .

- (a) For which values of  $P$  is the population increasing?

$$0 < P < 6000$$

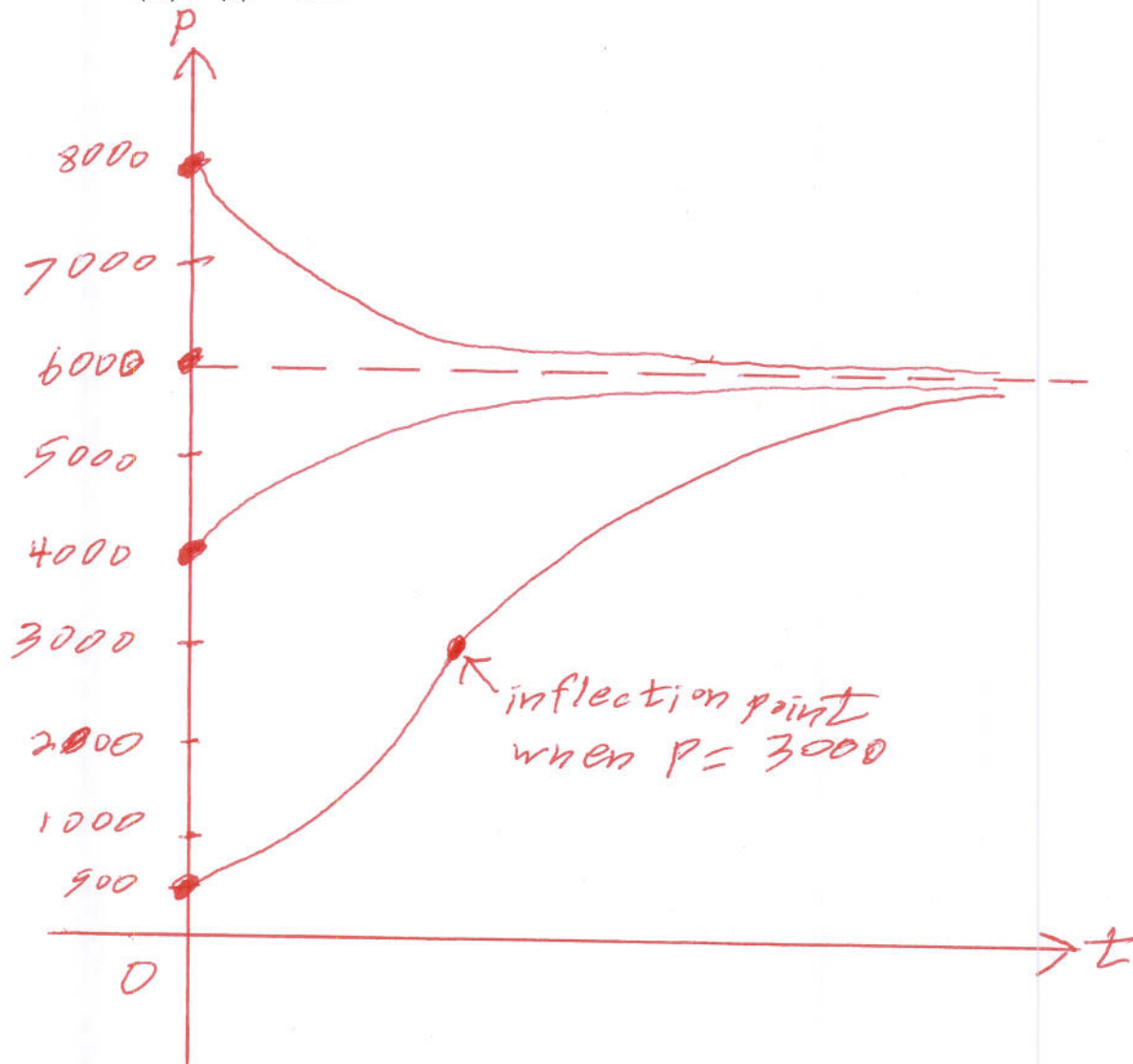
- (b) For which values of  $P$  is the population decreasing?

$$P > 6000 \text{ (also for } P < 0 \text{ but we ignore this since it is a population model)}$$

- (c) For which values of  $P$  is the population in equilibrium? Determine whether each of these equilibrium values is stable or unstable.

$$P = 6000 \text{ is a stable equilibrium value and } P = 0 \text{ is an unstable equilibrium value}$$

- (d) Sketch a rough graph of  $P$  given that: (i)  $P(0) = 500$ , (ii)  $P(0) = 4000$ , (iii)  $P(0) = 6000$ , (iv)  $P(0) = 8000$ .



4. Consider the population model

$$\frac{dP}{dt} = 0.3P \left(1 - \frac{P}{200}\right) \left(\frac{P}{50} - 1\right)$$

where  $P$  is the population at time  $t$ .

- (a) For which values of  $P$  is the population increasing?

$50 < P < 200$  (also for  $P < 0$  but we ignore this since it is a population model)

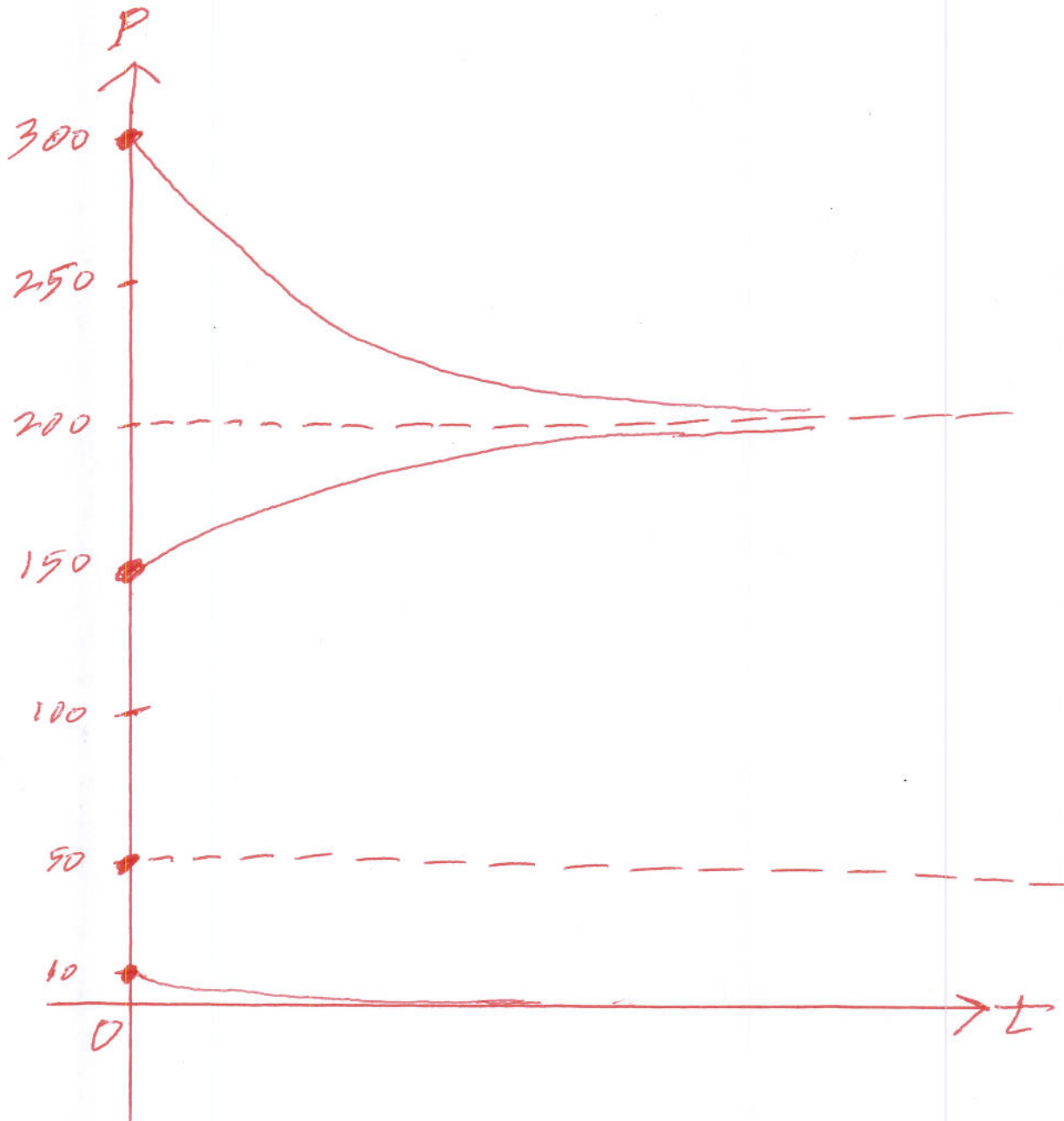
- (b) For which values of  $P$  is the population decreasing?

$P > 200$  or  $0 < P < 50$

- (c) For which values of  $P$  is the population in equilibrium? Determine whether each of these equilibrium values is stable or unstable.

$P = 0$  and  $P = 200$  are stable equilibrium values and  $P = 50$  is an unstable equilibrium value

- (d) Sketch a rough graph of  $P$  given that: (i)  $P(0) = 10$ , (ii)  $P(0) = 50$ , (iii)  $P(0) = 150$ , (iv)  $P(0) = 300$ .



5. Find all equilibrium values for the following differential equation. There is no need to discuss whether or not these equilibrium values are stable.

$$\frac{dP}{dt} = 0.5(P - 4)(P + 3)(P^2 - 25)$$

$$0 = 0.5(P - 4)(P + 3)(P - 5)(P + 5)$$

Equilibrium values are  $P = 4, -3, 5, \text{ and } -5$

6. Suppose  $y$  is a function of  $t$  which satisfies the differential equation below.

$$\frac{dy}{dt} = 0.25(y - 10)(20 - y)$$

Sketch plausible graphs for  $y$  as a function of  $t$  given each initial value below. Your graphs should clearly show if the  $y$ -values approach any particular values (i.e. horizontal asymptotes). You should draw all five graphs together on one set of coordinate axes.

(a)  $y(0) = 25$



(b)  $y(0) = 20$



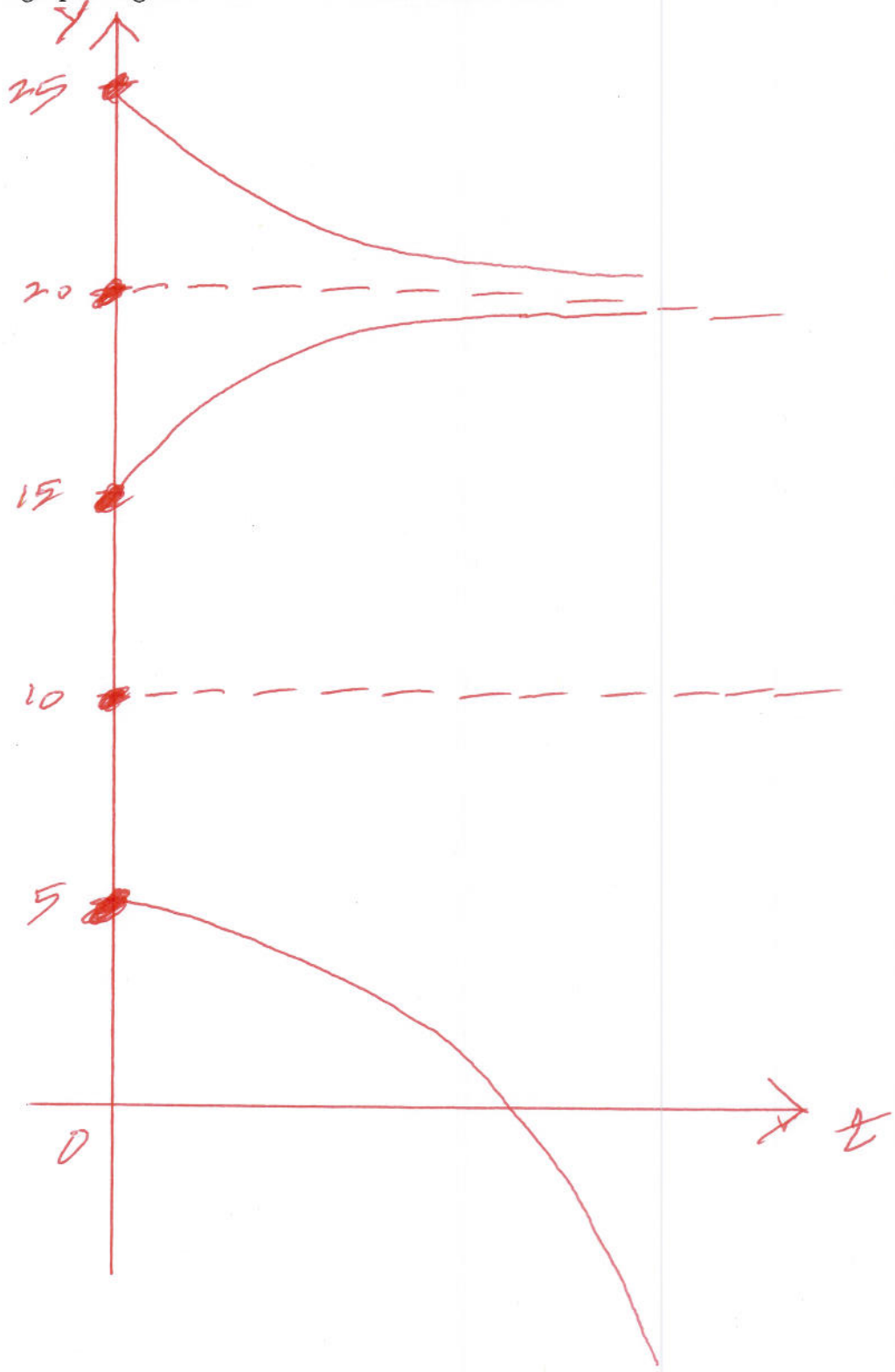
(c)  $y(0) = 15$



(d)  $y(0) = 10$



(e)  $y(0) = 5$



### 3 Important Growth Models

- LINEAR GROWTH ( $m = \text{slope}$ ,  $P_0 = \text{initial value}$ )

| Type       | Linear Model        | Explicit Solution |
|------------|---------------------|-------------------|
| continuous | $\frac{dP}{dt} = m$ | $P = mt + P_0$    |
| discrete   | $P(t) = P(t-1) + m$ | $P = mt + P_0$    |

- EXPONENTIAL GROWTH ( $r = \text{growth rate}$ ,  $P_0 = \text{initial value}$ )

| Type       | Exponential Model         | Explicit Solution |
|------------|---------------------------|-------------------|
| continuous | $\frac{dP}{dt} = rP$      | $P = P_0 e^{rt}$  |
| discrete   | $P(t) = P(t-1) + rP(t-1)$ | $P = P_0 (1+r)^n$ |

- LOGISTIC GROWTH ( $r = \text{growth rate}$ ,  $k = \text{carrying capacity}$ ,  $P_0 = \text{initial value}$ )

| Type       | Logistic Model  |
|------------|---|
| continuous | $\frac{dP}{dt} = rP \left(1 - \frac{P}{k}\right)$           |
| discrete   | $P(t) = P(t-1) + rP(t-1) \left(1 - \frac{P(t-1)}{k}\right)$ |

7. If  $P$  represents some population  $t$  years from now, and  $\frac{dP}{dt} = 20$ , then which of the following statements is correct?
- $P$  grows linearly by 20 people per year.
  - $P$  grows linearly by 120 people per year.
  - $P$  grows exponentially by 20% per year.
  - $P$  grows logistically with a carrying capacity of 20.
8. If  $P$  represents some population  $t$  years from now, and  $P(t) = 1.2P(t - 1)$ , then which of the following statements is correct?
- $P$  grows linearly by 20 people per year.
  - $P$  grows linearly by 120 people per year.
  - $P$  grows exponentially by 20% per year.
  - $P$  grows logistically with a carrying capacity of 120.
9. The population of a city was 5000 in 1980. Since then the population has been increasing by 100 people per year.
- Determine a discrete dynamical system with initial value to model the city's population.  
 $P(t) = P(t - 1) + 100$  and  $P(0) = 5000$
  - Determine a differential equation with initial value to model the city's population.  
 $\frac{dP}{dt} = 100$  and  $P(0) = 5000$
  - Determine an explicit formula for the city's population.  
 $P = 100t + 5000$
  - What does your model predict for the city's population in the year 2000?  
7000
  - When does your model predict the population will have reached 10000?  
In the year 2030
10. An initial deposit of \$200 is made into an account with an annual interest rate of 3% compounded annually.
- Determine a discrete dynamical system with initial value to model the amount of money in this account.  
 $A(t) = 1.03A(t - 1)$  and  $A(0) = 200$
  - Determine an explicit formula for the amount of money in this account.  
 $A = 200(1.03)^t$
  - How much money will the account hold 8 years after the initial deposit?  
\$253.35
  - How long will it take until the balance in this account is \$500?  
31.0 years

11. An initial deposit of 200 is made into an account with an annual interest rate of 3% compounded continuously.

(a) Determine a differential equation with initial value to model the amount of money in this account.

$$\frac{dA}{dt} = 0.03A \quad \text{and} \quad A(0) = 200$$

(b) Determine an explicit formula for the amount of money in this account.

$$A = 200e^{0.03t}$$

(c) How much money will the account hold 8 years after the initial deposit?

$$\$254.25$$

(d) How long will it take until the balance in this account is \$500?

$$30.5 \text{ years}$$

12. Why did I suggest using a discrete dynamical system for problem (10), but a differential equation for problem (11)?

The interest was compounded annually for problem (10) but continuously for problem (11)

13. The number of fish in a pond was 300 in 1970. Since then the number of fish has been increasing by 5% per year.

(a) Determine a discrete dynamical system with initial value to model the number of fish in this pond.

If we let  $F(t)$  represent the number of fish  $t$  years after 1970, then

$$F(t) = 1.05F(t-1) \quad \text{and} \quad F(0) = 300$$

(b) Enter this system into your calculator to make a table of values for the number of fish in the pond each year from 1970 to 1980.

| $t$ | $F(t)$ |
|-----|--------|
| 0   | 300.00 |
| 1   | 315.00 |
| 2   | 330.75 |
| 3   | 347.29 |
| 4   | 364.65 |
| 5   | 382.88 |
| 6   | 402.03 |
| 7   | 422.13 |
| 8   | 443.24 |
| 9   | 465.40 |
| 10  | 488.67 |

(c) Determine an explicit formula for the number of fish in the pond.

$$F(t) = 300(1.05)^t$$

(d) When will the fish population reach 750?

$$18.8 \text{ years later}$$

14. The number of fish in a pond was 300 in 1970. Since then the number of fish has been increasing by 5% per year.

(a) Determine a differential equation with initial value to model the number of fish in this pond.

$$\frac{dF}{dt} = 0.05F \quad \text{and} \quad F(0) = 300$$

(b) Use Euler's Method with  $\Delta t = 1$  to make a table of values for the number of fish in the pond each year from 1970 to 1980.

Since  $\Delta t = 1$ , our table will have the same values as the discrete dynamical system shown in #13b.

(c) Determine an explicit formula for the number of fish in the pond.

$$F(t) = 300e^{0.05t}$$

(d) When will the fish population reach 750?

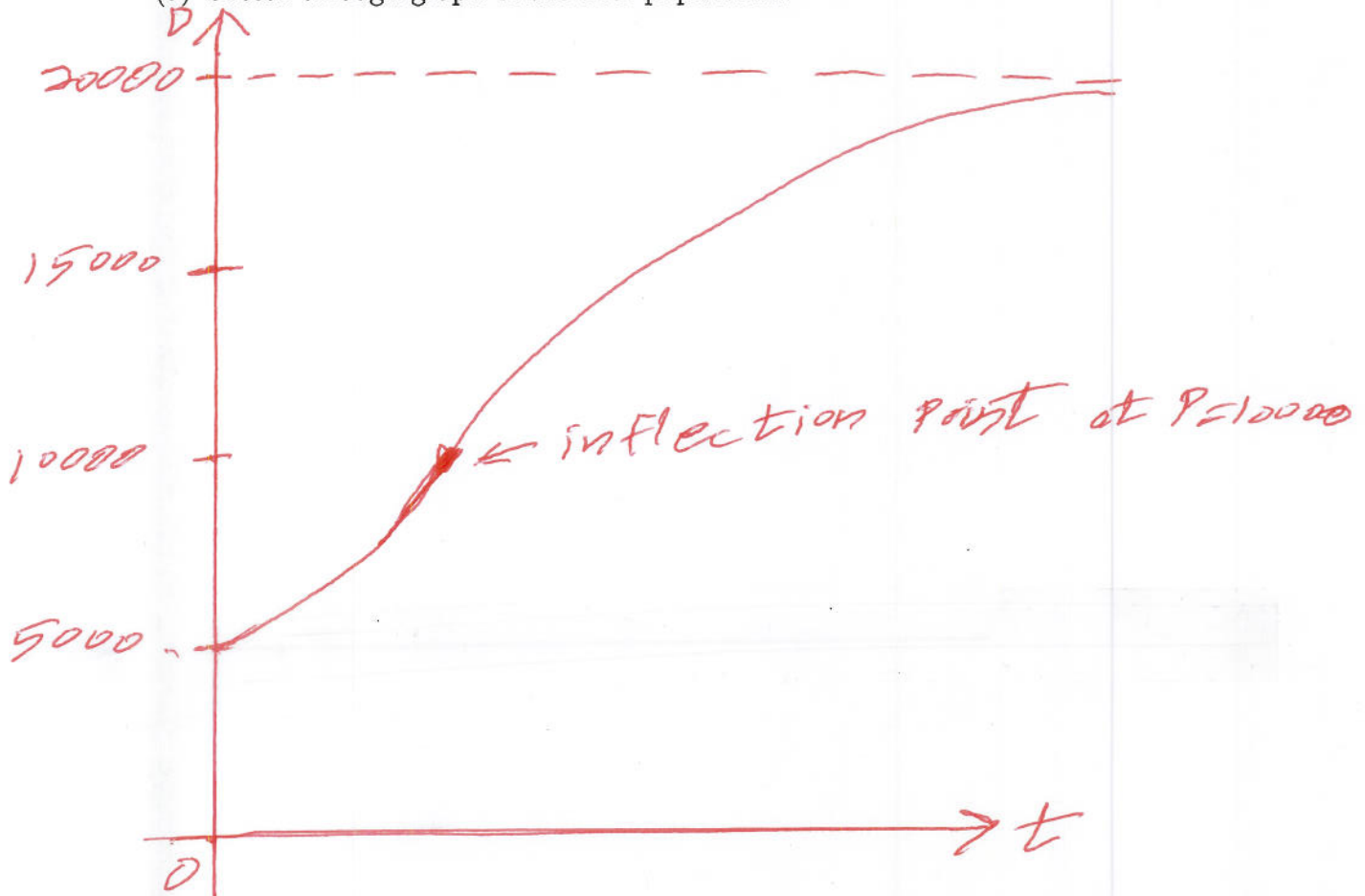
18.3 years later

15. Which model was best to use for the fish population – the discrete dynamical system or the differential equation? Why?

discussed in class

16. There are currently 5000 deer in a forest. Suppose the population of deer grows logistically with an intrinsic growth rate of 6% and a carrying capacity of 20,000.

(a) Sketch a rough graph of the deer population.



- (b) Determine a discrete dynamical system with initial value to model the deer population.  
If we let  $D(t)$  represent the number of deer  $t$  years from now, then

$$D(t) = D(t-1) + 0.06D(t-1) \left(1 - \frac{D(t-1)}{20000}\right) \quad \text{and} \quad D(0) = 5000$$

- (c) Make a table of values for the number of deer your discrete model predicts for the next 4 years.

| $t$ | $D(t)$ |
|-----|--------|
| 0   | 5000.0 |
| 1   | 5225.0 |
| 2   | 5456.6 |
| 3   | 5694.7 |
| 4   | 5939.1 |

- (d) Determine a differential equation with initial value to model the deer population.

$$\frac{dD}{dt} = 0.06D \left(1 - \frac{D}{20000}\right) \quad \text{and} \quad D(0) = 5000$$

- (e) Use Euler's Method with  $\Delta t = 1$  to make a table of values for the number of deer your continuous model predicts for the next 4 years.

| $t_{\text{current}}$ | $D_{\text{current}}$ | $D'_{\text{current}}$ | $D_{\text{next}} \approx D_{\text{current}} + D'_{\text{current}} \cdot \Delta t$ |
|----------------------|----------------------|-----------------------|---|
| 0                    | 5000.0               | 225.0                 | 5225.0  |
| 1                    | 5225.0               | 231.6                 | 5456.6  |
| 2                    | 5456.6               | 238.1                 | 5694.7  |
| 3                    | 5694.7               | 244.4                 | 5939.1  |
| 4                    | 5939.1               |                       |   |

Note that since  $\Delta t = 1$ , our table has the same values as the discrete dynamical system shown in part c.

- (f) Use Euler's Method with  $\Delta t = 0.5$  to make a table of values for the number of deer your continuous model predicts for the next 4 years.

| $t_{\text{current}}$ | $D_{\text{current}}$ | $D'_{\text{current}}$ | $D_{\text{next}} \approx D_{\text{current}} + D'_{\text{current}} \cdot \Delta t$ |
|----------------------|----------------------|-----------------------|---|
| 0.0                  | 5000.00              | 225.00                | 5112.50   |
| 0.5                  | 5112.50              | 228.34                | 5226.67   |
| 1.0                  | 5226.67              | 231.65                | 5342.50   |
| 1.5                  | 5342.50              | 234.92                | 5459.96   |
| 2.0                  | 5459.96              | 238.16                | 5579.04   |
| 2.5                  | 5579.04              | 241.37                | 5699.73   |
| 3.0                  | 5699.73              | 244.52                | 5821.99   |
| 3.5                  | 5821.99              | 247.63                | 5945.81   |
| 4.0                  | 5945.81              |                       |   |

- (g) How close are the approximations for discrete and continuous models?

discussed in class