

Math 500, Homework 2, due Wednesday, September 15

- (1) Prove that if H is a subgroup of index 2 of a group G , then H is normal in G .
- (2) Prove that if $n \geq 5$, the alternating group A_n has no subgroups of index k where $2 \leq k \leq n - 1$.
- (3) Prove that if G is a nonabelian group of order 6, then G is isomorphic to D_3 , the dihedral group of degree 3, as follows [this fact is proven in many books, so please do not look at the proofs there! just do it yourself]:
 - (a) Prove that G contains an element b of order 2.
 - (b) Prove that G contains an element a of order 3 by showing that if all elements of a group have order at most 2 then the group is abelian.
 - (c) Prove that $bab^{-1} = a^{-1}$ [Hint: why is $\langle a \rangle$ normal in G ?]
 - (d) Use the above to prove that there is an isomorphism $D_3 \rightarrow G$.
- (4) Prove that $\text{GL}_2(\mathbf{F}_2)$ is isomorphic to S_3 .
- (5) Prove that, for $n \geq 3$, every $\sigma \in A_n$ is a product of 3-cycles. [Hint: Show that $(1\ 2\ 3)$ and $(i\ j\ k)$ are conjugate by first treating the case when they are not disjoint (so the two permutations together move at most 5 letters) and then the case when they are disjoint.]
- (6) Prove that if a normal subgroup H of A_n contains a 3-cycle, then $H = A_n$. [Hint: For $n = 5$ this is Lemma 2.109 of Rotman.]
- (7) Let $\text{Aut}(G, X)$ denote the set of all actions of G on X , that is, all functions $G \times X \rightarrow X$ satisfying the axioms for a group action. Construct a bijection

$$\text{Hom}_{\mathbf{Groups}}(G, \text{Symm}(X)) \longrightarrow \text{Act}(G, X)$$

such that, if $H \rightarrow G$ is a homomorphism of groups and X is a G -set, there is a natural commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{Groups}}(G, \text{Symm}(X)) & \longrightarrow & \text{Act}(G, X) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathbf{Groups}}(H, \text{Symm}(X)) & \longrightarrow & \text{Act}(H, X). \end{array}$$

- (8) Prove that if H is a normal subgroup of a group G such that H and G/H are finitely generated, then G is finitely generated. If H and G/H are finitely presented, must G/H be finitely presented?
- (9) Suppose σ is an element of S_n . Find and prove a formula for the size of the conjugacy class of σ in S_n . [Hint: first write σ as a product of disjoint cycles.]