Math 500, Homework 4, due October 18

(1) Construct a nonabelian group of order 55.

(2) Let R be the set of 2×2 matrices over $\mathbf{C}$ of the form

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix},$$

where $\overline{z}, \overline{w}$ are the complex conjugates of z, w. Prove that R is a ring under matrix multiplication and that every nonzero element of R has a multiplicative inverse. Prove that R is isomorphic to the ring $\mathbb{Q}$ whose underlying set is a 4-dimensional real vector space with basis vectors $\mathbf{1}, i, j, k$, in which addition is the vector space addition, and multiplication is $\mathbf{R}$-linear and distributive and satisfies $\mathbf{1} \cdot x = x$ for all $x \in \mathbb{Q}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $ik = -ki = -j$, $jk = -kj = -i$.

(3) Prove that $\{a + b\sqrt{2} \ : \ a, b \in \mathbf{Q}\}$ is a field.

(4) An element x of a ring R is *nilpotent* if $x^n = 0$ for some $n \in \mathbf{N}$. Prove that the set N of all nilpotent elements in a commutative ring forms an ideal. Prove that the quotient R/N is a ring with no nilpotent elements.

(5) Prove that, if $I \subset R$ is an ideal, then the set of prime ideals of R/I is in bijective correspondence with the set of prime ideals of R that contain I.

(6) What are the units in the ring of Gaussian integers $\mathbf{Z}[i]$?

(7) A (commutative, with one) ring R *has dimension* n if, for every chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_k \subset R$ with $P_i \neq P_{i+1}$ for all i, one has $k \leq n$. Prove that $\mathbf{C}[x]/(x^m)$ has dimension 0 for every $m \geq 1$, and that $\mathbf{C}[x]$ has dimension 1.

(8) Prove that a UFD R is a PID if and only if every nonzero prime ideal is maximal.

(9) Prove that the ideal $(x^2 - y, z)$ in $\mathbf{Q}[x, y, z]$ is a proper ideal.

This problem is optional! Let X and Y be topological spaces, and let $\phi : X \to Y$ be a continuous function. Prove that the function $\phi^* : \mathcal{C}(Y, \mathbf{C}) \to \mathcal{C}(X, \mathbf{C})$, from the ring of continuous complex-valued functions on Y to the ring of such functions on X, given by $\phi^*(f) = f \circ \phi$, is a ring homomorphism. If X and Y are compact Hausdorff spaces, and Q is a maximal ideal of $\mathcal{C}(X, \mathbf{C})$, what is $(\phi^*)^{-1}(Q)$?