

Math 500, Homework 7, due November 17

- (1) Describe the prime ideals in $\mathbf{R}[x]$. Describe the prime ideals in $\mathbf{C}[x]$. Explain why it is difficult to list the prime ideals in $\mathbf{Q}[x]$.
- (2) Describe the set of units in the ring $\mathbf{Z}[\sqrt{-p}]$ where $p > 1$ is a prime integer.
- (3) Prove that the units in the ring $\mathbf{Z}[\sqrt{7}]$ are the elements of the form $a + b\sqrt{7}$ where $a^2 - 7b^2 = \pm 1$. Show that $8 + 3\sqrt{7}$ is a unit in $\mathbf{Z}[\sqrt{7}]$.
- (4) Let $\mathbf{Z}[\sqrt{7}]^\times$ denote the set of units in $\mathbf{Z}[\sqrt{7}]$. Define a function $\phi : \mathbf{Z}[\sqrt{7}]^\times \rightarrow \mathbf{R}^2$ by $\phi(a + b\sqrt{7}) = (\log |a + b\sqrt{7}|, \log |a - b\sqrt{7}|)$. Prove that it is a group homomorphism (where \mathbf{R}^2 has its structure as an additive group). Then prove that

$$\text{Im}(\phi) \subset \{(x, y) \in \mathbf{R}^2 : x + y = 0\} \cong \mathbf{R}.$$

Next, prove that the set of nonzero elements in $\text{Im}(\phi) \subset \mathbf{R}$ contains a smallest element. Use this to prove that $\text{Im}(\phi)$ is an infinite cyclic group. Show that $\Phi(8 + 3\sqrt{7})$ is a generator of this group. Conclude that

$$\mathbf{Z}[\sqrt{7}]^\times = \{\pm(8 + 3\sqrt{7})^k : k \in \mathbf{Z}\}.$$

- (5) Let $f \in F[x]$ be an irreducible polynomial of degree n (F is a field), and let E/F be a splitting field of f over F . Prove that n divides $[E : F]$. Prove that if f is separable, then n divides $|\text{Gal}(E/F)|$.
- (6) Let $f \in F[x]$, let E/F be a splitting field. Prove that if f is irreducible over F , then $\text{Gal}(E/F)$ acts transitively on the set of roots of f .
- (7) Do exercises 82, 83, 84, and 85 of Rotman's *Galois Theory* (page 75). Using your solutions, write out a complete proof of Lemma 73 of Rotman (page 72).