Math 500, Homework 8, due December 8

(1) Recall that a field F is *perfect* if every polynomial in $F[x]$ is separable.
   (a) Prove that a field of characteristic $p > 0$ is perfect if and only if $x^p - a$ has a root in F for all $a \in F$.
   (b) Prove that $\mathbf{F}_p(x)$, the field of rational functions with coefficients in the field $\mathbf{F}_p$ with p elements, is not perfect.

(2) Prove that A_4 has no subgroup of order 6. Use this to find a quartic irreducible polynomial $f(x)$ in $\mathbb{Q}[x]$ whose splitting field $E/\mathbb{Q}$ has an intermediate extension $\mathbb{Q} \subset K \subset E$ with the properties that
   (a) $[K : \mathbb{Q}] = 4$, but
   (b) there is no intermediate extension $\mathbb{Q} \subset L \subset K$ with $[L : \mathbb{Q}] = 2$.

(3) Prove that if $f \in \mathbb{Q}[x]$ is a quartic polynomial, then its discriminant is the discriminant of its resolvent cubic.

(4) Show that $\mathbf{F}_p(x, y)$ is a finite extension of $\mathbf{F}_p(x^p, y^p)$ but it is not a simple extension.

(5) Prove that if E/F is Galois and $F \subset B \subset E$ is an intermediate extension then E/B is Galois.

(6) Let $E = \mathbb{Q}(x_1, \ldots, x_n)$ with S_n acting by permuting the variables.
   (a) Check that S_n is acting by field automorphisms. Prove that the fixed field $F = \mathbb{Q}(x_1, \ldots, x_n)^{S_n}$ satisfies
      (i) $[E : F] = n!$ and
      (ii) $\mathrm{Gal}(E/F) = S_n$.
   (b) Prove that every finite group occurs as the Galois group of some field extension L/K.

   Remark. It is an open problem whether every finite group can be realized as the Galois group of some field extension $E/\mathbb{Q}$.

(7) Let E/F be a Galois extension and let $p \in F[x]$ be irreducible. Prove that all the irreducible factors of p in $E[x]$ have the same degree.