Math 428, First Midterm Name Fred Gauss

Problem 1. (5 +5 points)
Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Give the definition of $V(I)$.

$$V(I) = \{ a \in \mathbb{A}^n_k \mid f(a) = 0 \ \forall f \in I \}$$

Let $V \subset \mathbb{A}^n_k$ be a subset. Give the definition of $I(V)$.

$$I(V) = \{ f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \ \forall a \in V \}.$$

Problem 2. (8 points) State Hilbert's Nullstellensatz.

Assume k is algebraically closed. Then for all ideals $I \subseteq k[x_1, .., x_n]$,

$$I(V(I)) = \sqrt{I}.$$

Problem 3. (10 points) Give a description of all the prime ideals of $\mathbb{C}[x, y]$. Give (some) justification for your answer.

maximal ideals: $(x-a, y-b)$ for points $(a,b) \in \mathbb{C}^2$.

also : $\{0\}$ and

(f) for irreducible polynomials $f \in \mathbb{C}[x,y]$.

Justification: we proved that $(x-a, y-b)$ is maximal and we know the others are prime. If $I \subseteq \mathbb{C}[x,y]$ is prime, Hilbert's Nullstellensatz says $I(V(I)) = I$, and we classified the irred. subsets of $\mathbb{A}^2$ in class — they are points, $\emptyset$, $\mathbb{A}^2$, and irred. curves. So these are the only ideals that can arise.

Problem 4. (12 points) Prove that the plane cubic curve

$$V\big(y^2 - x(x-1)(x-\lambda)\big) \subset \mathbb{A}^2_{\mathbb{C}}$$

is irreducible for every choice of $\lambda \in \mathbb{C}$.

We will prove $f(x,y) = y^2 - x(x-1)(x-\lambda)$ is irred. It then follows that (f) is prime, so by Nullstellensatz $I(V(f)) = (f)$, and then $V(f)$ is irreducible.

Think of $f \in \mathbb{C}[x][y]$, and suppose $f = g_1 \cdot g_2$. If the y-degree $\deg_y g_1$ of g_1 is zero, i.e. $g_1 \in \mathbb{C}[x]$, and $g_2 = a(x)y^2 + b(x)y + c(x)$, then $g_1 \cdot a = 1$, so g_1 is a constant. Thus we may assume $\deg_y(g_1) \geq 1$, and by symmetry $\deg_y(g_1) = \deg_y(g_2) = 1$, say $g_1 = a_1(x)y + b_1(x)$, $g_2 = a_2(x)y + b_2(x)$.

Then $a_1(x)a_2(x) = 1$ so a_1, a_2 are constants, which we may rescale to be 1.

Also $a_1b_2 + a_2b_1 = b_2 + b_1 = 0$

$\implies b_2 = -b_1$, and $-b_1^2 = -x(x-1)(x-\lambda)$.

Since $x(x-1)(x-\lambda)$ is not a square, we are done.

Problem 5. (6+8 points)

(1) Prove that the subset

$$V = \{(t^2 - 1, t^3, t^4) \mid t \in \mathbb{C}\} \subset \mathbb{A}^3_{\mathbb{C}}$$

is an algebraic subset of $\mathbb{A}^3_{\mathbb{C}}$.

let $x' = x+1$. I claim

$\underbrace{V}_{LHS} = \underbrace{V(y^2-(x')^3}_{RHS}, z-(x')^2)$. It's clear that $LHS \subseteq RHS$. On the other hand, suppose $(x,y,z) \in RHS$. Let t be a square root of $x+1 = x'$. Then $z = (x+1)^2 = t^4$. Since also $y^2 = (x')^3 = t^6$, we have $y = \pm t^3$. If $y = t^3$ then $(x,y,z) = (t^2,t^3,t^4)$; if $y = -t^3$, then $(x,y,z) = ((-t)^2,(-t)^3,(-t)^4)$.

So $RHS \subseteq LHS$.

(2) Is V irreducible? Justify your answer.

We have a surjective polynomial map

$$\mathbb{A}^1_{\mathbb{C}} \xrightarrow{\varphi} V \quad \text{given by } \varphi(t) = (t^2, t^3, t^4).$$

If $W_1 \xrightarrow{\Phi} W_2$ is a surjective map of sets, then $\Phi^* : \mathrm{Fun}(W_2,k) \to \mathrm{Fun}(W_1,k)$ is injective: if $f_1\circ\Phi = \Phi^* f_1 = \Phi^* f_2 = f_2\circ\Phi$, then $f_1(\Phi(a)) = f_2(\Phi(a)) \ \forall a \in W_1 \Rightarrow f_1(b) = f_2(b) \ \forall b \in \mathrm{Im}(\Phi) = W_2$
$\Rightarrow f_1 = f_2$.

So now, suppose $f \in k[x_1,..,x_n]$ and $f\circ\varphi = f|_V \circ \varphi$ is the zero function. Then, since $f|_V\circ\varphi = \varphi^*(f|_V)$, we have $f|_V = 0$, i.e. $f \in I(V)$. We get an induced injective homomorphism $k[x_1,..,x_n]/I(V) \xrightarrow{\Phi} k[t]$. So $k[x_1,..,x_n]/I(V)$ is a domain $\Rightarrow$ V is irred.

Problem 6. (8 + 8 points) Let $V \subset \mathbb{A}_k^n$ be a variety, let $\overline{f} \in k[x_1, \dots, x_n]$ and let $f \in \Gamma(V)$ be its restriction to V. Let

$$\operatorname{graph}(f) = \{(a_1, \dots, a_n, a_{n+1}) \in \mathbb{A}_k^{n+1} \mid (a_1, \dots, a_n) \in V \text{ and } a_{n+1} = f(a_1, \dots, a_n)\}.$$

(1) Show that graph(f) is an algebraic subset of $\mathbb{A}_k^{n+1}$.

Let $F(x_1,..,x_{n+1}) = x_{n+1} - \overline{f}(x_1,..,x_n)$

Then $\operatorname{graph}(f) = \left\{ (a_1,..,a_{n+1}) \in \mathbb{A}_k^{n+1} \;\middle|\; \begin{array}{l} g(a_1,..,a_n) = 0 \ \forall g \in I(V) \\ \text{and } F(a_1,..,a_{n+1}) = 0 \end{array} \right\}$

$= V(I(V) \cup \{F\})$.

(2) Prove that there is an isomorphism $G : V \to \operatorname{graph}(f)$: that is, such that G is a polynomial map and there is a polynomial map $p : \operatorname{graph}(f) \to V$ such that $p \circ G = \mathrm{id}$ and $G \circ p = \mathrm{id}$.

Define G by $G(a_1,..,a_n) = (a_1,..,a_n, f(a_1,..,a_n))$. Define $\tilde{p} : \mathbb{A}_k^{n+1} \longrightarrow \mathbb{A}_k^n$ by $\tilde{p}(a_1,..,a_{n+1}) = (a_1,..,a_n)$. Let $p = \tilde{p}|_{\operatorname{graph}(f)}$. Then $\forall (a_1,..,a_n, f(a_1,..,a_n))$ in graph(f), $p(a_1,..,a_n, f(a_1,..,a_n)) = (a_1,..,a_n) \in V$. So p gives a polynomial map from graph(f) to V.

If $(a_1,..,a_n) \in V$, then

$p(G(a_1,..,a_n)) = p(a_1,..,a_n, f(a_1,..,a_n)) = (a_1,..,a_n)$, so $p \circ G = \mathrm{id}$. If $(a_1,..,a_n, f(a_1,..,a_n)) \in \operatorname{graph}(f)$, then $(G \circ p)(a_1,..,a_n, f(a_1,..,a_n)) = G(a_1,..,a_n) = (a_1,..,a_n, f(a_1,..,a_n))$. So $G \circ p = \mathrm{id}$. Thus G is an isomorphism

L hf] d_e 6.)/ . `_Z^cb* CVc T 7 T )b* SV R^ ZaaVUdTZS\V Yh` VabdaWRTV Z^ B^n_D)Z^ `RacZTd\Ra, b 2 C\[p_1<; ; ; <p_n OZb ^_^i Va_*- GY_f cYRc cYVaV U_Vb ^_c VgZbc R^ ZaaVUdTZS\V R\XVSaRZT bVc U bdTY cYRc T ⊂ U ⊂ B^n_D f ZcY T 5 U R^U U 5 B^n_D-

Note first that if P is a prime ideal and $g \in P$ then P contains an irreducible factor of g. Applied to $I = I(V) = I(V(f))$ we have $f \in I$, so some irred. factor F of f lies in I. Now $(f) \subseteq (F)$

$$\Rightarrow I(V(f)) \subseteq I(V(F)) \underset{\substack{\uparrow \\ \text{by Hilbert}}}{=} \sqrt{(F)} = (F),$$

so $(F) = I(V(f))$ and $V = V(F)$. So we may assume f is irreducible.

Now suppose $V \subseteq W \subseteq \mathbb{A}^n_{\mathbb{C}}$ with $W \neq \mathbb{A}^n_{\mathbb{C}}$, irreducible. Then $\{0\} \neq I(W)$ is prime, so it contains an irred. elt. g. Now $g \in I(W) \subseteq I(V) = (f) \Rightarrow f$ divides g. Both are irreducible, though, so they are associates, i.e. $(f) = (g) \Rightarrow$

$$V = V(f) = V(g) = W.$$