Math 428, First Midterm Name_____

Problem 1. (5 +5 points)
Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Give the definition of $V(I)$.

Let $V \subset \mathbb{A}^n_k$ be a subset. Give the definition of $I(V)$.

Problem 2. (8 points) State Hilbert's Nullstellensatz.

Problem 3. (10 points) Give a description of all the prime ideals of $\mathbb{C}[x, y]$. Give (some) justification for your answer.

Problem 4. (12 points) Prove that the plane cubic curve

$$V\big(y^2 - x(x-1)(x-\lambda)\big) \subset \mathbb{A}^2_{\mathbb{C}}$$

is irreducible for every choice of $\lambda \in \mathbb{C}$.

Problem 5. (6+8 points)

(1) Prove that the subset

$$V = \{(t^2 - 1, t^3, t^4) \mid t \in \mathbb{C}\} \subset \mathbb{A}^3_{\mathbb{C}}$$

is an algebraic subset of $\mathbb{A}^3_{\mathbb{C}}$.

(2) Is V irreducible? Justify your answer.

Problem 6. (8 + 8 points) Let $V \subset \mathbb{A}_k^n$ be a variety, let $\overline{f} \in k[x_1, \ldots, x_n]$ and let $f \in \Gamma(V)$ be its restriction to V. Let

$$\text{graph}(f) = \{(a_1, \ldots, a_n, a_{n+1}) \in \mathbb{A}_k^{n+1} \mid (a_1, \ldots, a_n) \in V \text{ and } a_{n+1} = f(a_1, \ldots, a_n)\}.$$

(1) Show that $\text{graph}(f)$ is an algebraic subset of $\mathbb{A}_k^{n+1}$.

(2) Prove that there is an isomorphism $G : V \to \text{graph}(f)$: that is, such that G is a polynomial map and there is a polynomial map $p : \text{graph}(f) \to V$ such that $p \circ G = \text{id}$ and $G \circ p = \text{id}$.

Problem 7. (10 points) Let $V = V(f)$ be an irreducible hypersurface in $\mathbb{A}^n_{\mathbb{C}}$ (in particular, $f \in \mathbb{C}[x_1, \ldots, x_n]$ is nonzero). Show that there does not exist an irreducible algebraic set W such that $V \subset W \subset \mathbb{A}^n_{\mathbb{C}}$ with $V \neq W$ and $W \neq \mathbb{A}^n_{\mathbb{C}}$.