1. Recall that

$U_i = \{(x_0 : \dots : x_n) \in \mathbb{P}^n_K \mid x_i \neq 0\}$, and we

have a bijection $\varphi_i : \mathbb{A}^n_K \to U_i$ by

$$\varphi_i(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) = (a_0 : \dots : a_{i-1} : 1 : a_{i+1} : \dots : a_n).$$

(a) Prove that if

$V = V(f_1, \dots, f_r) \subseteq \mathbb{P}^n_K$ is an algebraic subset, then $\varphi_i(\overline{V}) = V \cap U_i$, where

$$\overline{V} = V(\varphi_i^* f_1, \dots, \varphi_i^* f_r) \subseteq \mathbb{A}^n_K \text{ and}$$

$$\varphi_i^* f_k = f_k(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n).$$

(b) Prove that if $\overline{V}$ is reducible (for some i) then V is reducible.

(c) Give an example in which $\overline{V}$ is irreducible (say for $i=0$) but V is reducible.

(d) [a bit harder] Show that if, for every i, $\bar{V}$ is irreducible, then V is irreducible.

2. Suppose $f_d, f_{d+1} \in \mathbb{C}[x,y]$ are homogeneous of degrees d, $d+1$ respectively. Suppose $f_d + f_{d+1}$ is <u>not</u> irreducible.

(a) Give a complete description of its irreducible factors. (b) Prove that $V(f_d + f_{d+1})$ is a union of irreducible components, each of which can be rationally parametrized.

3. Let $p = (x : y : z) \in \mathbb{P}^2_{\mathbb{C}}$.

(a) Show that
$$\{(a,b,c) \in \mathbb{C}^3 \mid ax + by + cz = 0\}$$
is a linear subspace of dimension 2.

(b) Show that for any finite set of points in $\mathbb{P}^2_{\mathbb{C}}$, there is a line not passing through any of them.

4. For the purposes of this problem, a <u>conic</u> $C \subseteq \mathbb{P}^2_{\mathbb{C}}$ is a subset of the form

$$C = V(F) \text{ where } F \text{ is a}$$

nonzero homogeneous polynomial of degree 2. Suppose $p_1, \ldots, p_4 \in \mathbb{P}^2_{\mathbb{C}}$ are distinct points in $\mathbb{P}^2_{\mathbb{C}}$. Must there be a conic that passes through all four points?

5. Let $C = V(xy^4 + yz^4 + xz^4) \subseteq \mathbb{P}^2_{\mathbb{C}}$.

(a) Show that C is irreducible.

(b) For each i, compute the multiple points and tangent lines of $\varphi_i^{-1}(C \cap U_i)$.

6. Is $C = V(x^n + y^n + z^n) \subseteq \mathbb{P}^2_{\mathbb{C}}$ irreducible $(n \geq 1)$? Does it have multiple points?