Math 428, Homework 2

Problem 1. Prove that if k is a field then k has only two ideals. Prove that k is noetherian. Conclude that for $n \geq 1$, $k[x_1, \dots, x_n]$ is a noetherian ring.

Problem 2. * Describe an infinite *descending* chain of ideals in $k[x]$ where k is a field: that is, a list of ideals with inclusions,

$$\dots I_k \subset I_{k-1} \subset \dots \subset I_1 \subset k[x]$$

where $I_k \neq I_{k+1}$ for all $k \geq 1$.

Problem 3. Prove that if $S = \{s_1, \dots, s_n\} \subset R$ is a ring, then the ideal (S) generated by S may also be described as:

$$(S) = \{a_1 s_1 + \dots a_n s_n \mid a_1, \dots, a_n \in R\}.$$

Problem 4. * Let $I \subset R$ be an ideal in a ring R. Prove that R/I is an integral domain if and only if I is a prime ideal. Prove that R/I is a field if and only if I is a maximal ideal. Prove that every maximal ideal is prime.

Problem 5. * Consider the ideal I in $\mathbb{Z}[x]$ generated by the two polynomials $f(x) = 2x^2 + 1$ and $g(x) = 3x^3 + 1$. Prove that $I \neq \mathbb{Z}[x]$. [Hint: there is no general theorem to appeal to here, I think.]

Problem 6. Suppose that $R_k, k \geq 1$ are rings (commutative, with 1) and that $\phi_k : R_k \to R_{k+1}$ is an injective homomorphism for each k; by slight abuse of notation, we think of R_k as being a subring of R_{k+1}.

(1) Prove that the set $R = \cup_{k \geq 1} R_k$ has a unique structure of ring that makes each R_k a subring of R.

(2) Define the polynomial ring $R = k[x_1, x_2, \dots]$ in infinitely many variables by $R = \cup_n k[x_1, \dots, x_n]$. Give an infinite ascending chain of ideals in R that is *not* stationary—that is, a chain

$$I_1 \subset I_2 \subset \dots \subset R$$

such that $I_k \neq I_{k+1}$ for all $k \geq 1$.

Problem 7. * Consider the ring $\mathbb{C}[x, y]$. Let $m = (x, y)$, the ideal generated by x and y.

(1) Prove that this is exactly the ideal of polynomials $f(x, y)$ for which $f(0, 0) = 0$.

(2) For $k \geq 1$, let $m^k = (x^k, x^{k-1}y, \dots, xy^{k-1}, y^k)$. Prove that $V = \mathbb{C}[x, y]/m^{k+1}$ is a finite-dimensional complex vector space with a basis consisting of all cosets $x^u y^v + m^{k+1}$ of monomials $x^u y^v$ of degree less than $k+1$ (i.e. with $u + v < k+1$). Prove that m^k/m^{k+1} is exactly the span in V of the vectors (written in coset form)

$$x^k + m^{k+1}, x^{k-1}y + m^{k+1}, \dots, xy^{k-1} + m^{k+1}, y^k + m^{k+1}.$$

(3) Use the previous part of the problem to prove that the ideal m^k cannot have fewer than $k+1$ generators. [Hint: if m^k could be generated by elements $v_1, \dots, v_\ell$, their images in $V = m^k/m^{k+1}$ would span V as a vector space—why? Draw a grid, with boxes representing the monomials, to help you see the answer.]