Math 428, Homework 3

Write up and turn in solutions to problems 1, 2, 4, 5.

Problem 1. Prove that every proper ideal in a Noetherian ring is contained in a maximal ideal.

Problem 2. Remind yourself of the definitions of a *prime element* and an *irreducible element* of a ring R (commutative with 1). Prove that if R is a unique factorization domain, then these definitions are equivalent. Prove that an element $x \in R$ is prime if and only if the ideal (x) is a prime ideal.

Problem 3. Let I be an ideal in a ring R. If $a^n \in I$ and $b^m \in I$ $(n, m > 0)$ prove that $(a+b)^{n+m} \in I$ and $(ab)^{\max(n,m)} \in I$. Prove that $\sqrt{I}$ is an ideal of R and in fact a radical ideal of R. Show that every prime ideal is a radical ideal. Give an example of an ideal in $\mathbb{C}[x]$ that is not prime but whose radical is prime.

Problem 4. Show that $I = (x^2 + 1) \subset \mathbb{R}[x]$ is a prime ideal (and hence is a radical ideal) but that I is not the ideal of any algebraic subset of $\mathbb{A}^1_{\mathbb{R}}$.

Problem 5. Suppose that R is an integral domain. Let $\mathbb{F}$ denote the set of ordered pairs (a, b) with $a, b \in R$ and $b \neq 0$. Define an equivalence relation on $\mathbb{F}$ by $(a, b) \equiv (c, d)$ if and only if $ad = bc$ in R.

(1) Prove that this really is an equivalence relation!

Now, let F denote the set of equivalence classes in $\mathbb{F}$ under this equivalence relation. We think of (a, b) as the fraction $\frac{a}{b}$. Accordingly, we define operations $+$ and $\cdot$ on F by the formulas $(a, b) + (u, v) = (av + bu, bv)$ and $(a, b) \cdot (u, v) = (au, bv)$.

(2) Prove that these give well-defined operations of addition and multiplication on F: that is, that if $(a, b) \equiv (c, d)$ and $(u, v) \equiv (s, t)$, then $(a, b) + (u, v) \equiv (c, d) + (s, t)$ and $(a, b) \cdot (u, v) \equiv (c, d) \cdot (s, t)$.

(3) Prove that, with these operations, F is a field, and that $\phi(a) = (a, 1)$ defines an injective ring homomorphism $\phi : R \to F$.

F is called the *field of fractions* of R. We usually write the equivalence class of (a, b) as $\frac{a}{b}$. Note: you have probably seen this construction before in the case $R = \mathbb{Z}$ and $F = \mathbb{Q}$.

Problem 6. Show that $f(x, y) = y^2 + x^2(x - 1)^2 \in \mathbb{R}[x, y]$ is an irreducible polynomial, but that $V(f)$ is reducible.