

Math 428, Homework 5

Problem 1. Fix a field k . Let $f_1, \dots, f_r, g \in k[x_1, \dots, x_n]$ be polynomials with g nonzero. Let $k(x_1, \dots, x_n)$ denote the field of rational functions in variables x_1, \dots, x_n (with coefficients in k). Let $R = k[x_1, \dots, x_n, g^{-1}]$ denote the subset of $k(x_1, \dots, x_n)$ consisting of rational functions that can be represented as fractions of the form $\frac{f}{g^N}$ for some $N \geq 0$ and some $f \in k[x_1, \dots, x_n]$.

- (1) Prove that $k[x_1, \dots, x_n, g^{-1}]$ is a subring of $k(x_1, \dots, x_n)$.
- (2) Construct an isomorphism from $Q = k[x_1, \dots, x_n, x_{n+1}]/(x_{n+1}g - 1)$ to $R = k[x_1, \dots, x_n, g^{-1}]$. [Hint: take x_{n+1} to $1/g$.]
- (3) Prove that the following are equivalent:
 - (a) There exist $a_1, \dots, a_r \in k[x_1, \dots, x_n]$ such that the image of $a_1f_1 + \dots + a_rf_r$ in Q is 1.
 - (b) There exist $c_1, \dots, c_r \in R$ such that $c_1f_1 + \dots + c_rf_r = 1$ in R .

Problem 2. Use the results of the previous problem to give a (somewhat) different proof of the Nullstellensatz (assuming, of course, that k is algebraically closed) as follows:

- (1) Suppose that $g \in V(f_1, \dots, f_r)$. Use the Weak Nullstellensatz to prove that the ideal in Q generated by the images of f_1, \dots, f_r is all of Q . [This is almost identical to what we did before.]
- (2) Use part (3) of the previous problem to conclude that there exist rational functions c_1, \dots, c_r in R such that $c_1f_1 + \dots + c_rf_r = 1$ in R .
- (3) Now, clear g from the denominators of c_1, \dots, c_r and conclude that there are a positive integer N and polynomials u_1, \dots, u_r such that $u_1f_1 + \dots + u_rf_r = g^N$ in $k[x_1, \dots, x_n]$.

Problem 3. Let $g \in k[x_1, \dots, x_n]$ be a nonzero polynomial. Let $V = V(x_{n+1}g - 1) \subset \mathbb{A}_k^{n+1}$. Define a function $\phi : V \rightarrow \mathbb{A}_k^n$ by $\phi(a_1, \dots, a_{n+1}) = (a_1, \dots, a_n)$. Prove that ϕ is injective and that its image is exactly $\mathbb{A}_k^n \setminus V(g)$.

Problem 4. Do the problem explained in the proof on the last page of the Week 4 notes (on the web).

Problem 5. Let $V = \mathbf{A}_{\mathbb{C}}^1$ (with coordinate t , so $\Gamma(V) = \mathbb{C}[t]$) and $W = V(y^2 - x^3) \subset \mathbf{A}_{\mathbb{C}}^2$.

- (a) Prove that the polynomial map $\Phi : V \rightarrow \mathbf{A}_{\mathbb{C}}^2$ given by $\phi(t) = (t^2, t^3)$ has W as its image, and that in fact this gives a bijection from V to W .

Let $\phi : V \rightarrow W$ be the induced polynomial map from V to W .

- (b) Prove that the induced map $\phi^* : \Gamma(W) \rightarrow \Gamma(V)$ is injective.
- (c) Identify the image of ϕ^* (note that it is a subring of $\mathbb{C}[t]$).