1. Prove that two curves in $\mathbb{P}^2_{\mathbb{C}}$ that do not share a common component intersect in only finitely many points.

[Hint: It may help to consider points of intersection on $\mathbb{A}^2_{\mathbb{C}} \subseteq \mathbb{P}^2_{\mathbb{C}}$ and on the line at infinity separately.]

2. Let $V = V(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2_{\mathbb{C}}$.

Let $\bar{x} = x + I(V)$, $\bar{y} = y + I(V) \in \Gamma(V)$.

Let $z = \frac{\bar{y}}{\bar{x}} \in \mathbb{C}(V)$. Find the the pole sets of z and z^2 (i.e. the subsets of V where z and z^2 are not defined).

3. Let V be a variety and $f \in \mathbb{C}(V)$.

Let $U = \{p \in V \mid f \text{ is defined at } p\}$.

Then f determines a function from U to $\mathbb{C}$.

Show that this function determines f uniquely.

4. Let $F(x,y) = y^2 - x(x-1)(x-\lambda) \in \mathbb{C}[x,y]$, and let $C = V(F) \subseteq \mathbb{A}^2_{\mathbb{C}}$.
Find a necessary and sufficient condition on λ under which, for all $p \in C$, either $\frac{\partial F}{\partial x}(p) \neq 0$ or $\frac{\partial F}{\partial y}(p) \neq 0$ (or both).

5. Let $C \subseteq \mathbb{A}^2_{\mathbb{C}}$ be an irreducible plane curve: that is, $C = V(F)$ for some irreducible $F \in \mathbb{C}[x,y]$. Suppose $\mathbb{C}(C) \cong \mathbb{C}(t)$. Prove that C admits a rational parametrization.

6. Quoting Spivak's "Calculus," "The world's sneakiest substitution is undoubtedly

$$t = \tan\left(\frac{\theta}{2}\right), \quad \theta = 2\arctan(t),$$

$$d\theta = \frac{2}{1+t^2}\,dt.$$

This substitution leads to the expressions

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$ "

(a) Explain what this substitution has to do with the rational parametrization of the circle that we discussed in class.

(b) Explain how this allows one to convert any integral that involves only $\sin(\theta)$ and $\cos(\theta)$, combined using addition, multiplication, and division, into the integral of a rational function.