# Linear Systems of Plane Curves

We are going to study more about plane curves for awhile. For the things we'll study, it's convenient to think ~~of~~ a plane curve as *being* a homogeneous polynomial $F \in k[x,y,z]_d$ (of degree $d$) up to nonzero scalar multiples. In other words, the *set* of all plane curves of degree $d$ will be

$$\mathbb{P}(k[x,y,z]_d) = \{0 \neq F \in k[x,y,z]_d\}/\sim$$

where $F \sim \lambda F$ for all $\lambda \in k^\times$. $(= k \setminus \{0\})$.

Remark

$$\dim_k(k[x,y,z]_d) = \dim_k k[x,y]_{\leq d}$$
$$= \sum_{m=0}^{d} (m+1) = \frac{(d+1)(d+2)}{2}.$$

Thus,

$$\dim \mathbb{P}(k[x,y,z]_d) = \frac{(d+1)(d+2)}{2} - 1 = \frac{d^2+3d}{2}$$

$$= \frac{d(d+3)}{2}$$

So,

"the plane curves of degree $d$ form a projective space of dimension $\frac{d(d+3)}{2}$."

Note that we are making a slightly unfamiliar use of "plane curves." Indeed, for us $x^2$ defines a plane curve of deg. 2, even though $V(x^2) = V(x)$ is a line. We think of $x^2$ as telling us a "double line."

The reason for doing this is that we want to study collections, or <u>families</u>, of curves.

<u>Def</u> The set of curves in a linear subspace $\mathbb{P}^k \subseteq \mathbb{P}(k[x,y,z]_d)$ is a <u>linear system</u> (or <u>linear series</u>) of curves of degree $d$.

<u>Example</u> The curves

$$C_{(a:b)} = V(ay^2z - x(x-z)(ax-bz))$$

form a linear system of plane cubics.

<u>Note</u> If you expand, you get

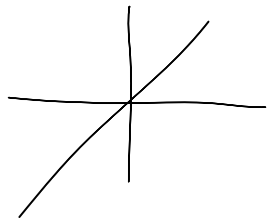
$$F_{(a:b)} = a(y^2z - x^3 + x^2z) + b(x^2z - xz^2).$$

The parameter $(a:b)$ lies on a $\mathbb{P}^1$;

when $a=1$ we get our usual family of irreducible plane cubics $y^2z - x(x-z)(x-bz)$.

When $a=0$, we get

$x^2z - xz^2 = xz(x-z)$, three lines in $\mathbb{P}^2$ meeting at ~~the~~ point $(0:1:0)$.



Example A line in $\mathbb{P}^2$ is given by

$ax+by+cz=0$. This corresponds to

$(a:b:c) \in \check{\mathbb{P}}^2 \leftarrow$ linear system of lines in $\mathbb{P}^2$.

Example A conic in $\mathbb{P}^2$ is given by

$ax^2+bxy+cxz+dy^2+eyz+fz^2=0$.

This corresponds to

$(a:b:c:d:e:f) \in \mathbb{P}^5$.

**Lemma** Let $p \in \mathbb{P}^2$ be a point. The set of curves of degree $d$ passing through $p$ forms a hyperplane in $\mathbb{P}^{d(d+3)/2}$.

Proof. "Evaluation at $p$" is linear: let $p = (a:b:c)$. Then the set of curves passing through $p$ is $\mathbb{P}(V)$ where

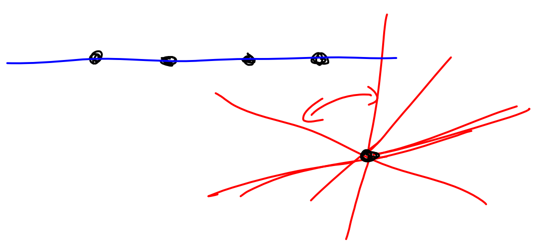
$$V = \ker\left(ev : k[x,y,z]_d \longrightarrow k\right),$$

$$ev(F) = F(a,b,c).$$ $\square$

**Cor** If $p_1, \ldots, p_k \in \mathbb{P}^2$ are distinct points, the set of curves of deg. $d$ passing through $p_1, \ldots, p_k$ is a linear subspace of dimension at least $\frac{d(d+3)}{2} - k$ (in particular, nonempty if this number is $\geq 0$).

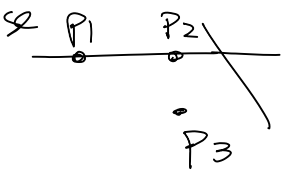
Proof. Each $p_i$ imposes a condition. The only question is whether they are independent conditions, i.e. whether they define linearly independent functionals on $k[x,y,z]_d$. $\square$.

Example Five points in $\mathbb{P}^2$ impose at most five conditions on conics. Since the space of conics is $\mathbb{P}^5$, there is always a conic through those five points. Might there be more than one? If four points are collinear, definitely!



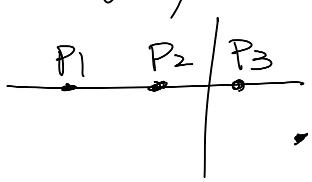
Prop Suppose $p_1, \dots, p_5 \in \mathbb{P}^2$, no four collinear. Then there is a unique conic passing through $p_1, \dots, p_5$.

Proof It suffices to show that not every conic containing $p_1, \dots, p_i$ passes through $p_{i+1}$, $i = 1, 2, 3, 4$. For $i=1$ this is clear. For $i=2$, if $p_1, p_2, p_3$ are not collinear then use



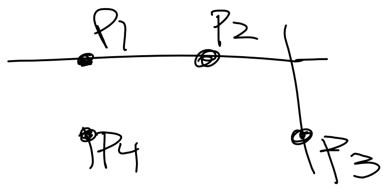

. If they are collinear, use an irreducible conic containing $p_1$ and $p_2$.

For $i=3$, if $p_1, p_2, p_3$ are collinear use




. If they are not, but $p_4$ lies on $\overline{p_1 p_2}$ (= line through $p_1, p_2$), use an irred. conic through $p_1, p_2, p_3$ (see below)

If $p_4$ does not lie on $\overline{p_1 p_2}$, use




$i = 4$ gets complicated... $\square$.

**Prop** Suppose $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$. If no three are collinear, there is a projective transformation of $\mathbb{P}^2$ taking them to

$$(1:0:0), \quad (0:1:0), \quad (0:0:1), \quad (1:1:1).$$

Pf. "No three collinear" implies, letting

$$p_i = [v_i], \quad v_i \in k^3, \text{ that } v_1, v_2, v_3$$

span $k^3$ and $p_4 = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$ with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$. Rescaling $v_1, v_2, v_3, v_4$, we may assume

$$v_4 = v_1 + v_2 + v_3.$$

Now our proj. transformation takes $v_i$ to $e_i$, $i = 1, 2, 3$. $\square$

<u>Corollary</u> If $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$, no 3 collinear, then there are irreducible conics through $p_1, p_2, p_3, p_4$.

Proof By the Prop, it suffices to check for $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, $(1:1:1)$.

This is an exercise. $\square$