<u>Thm</u> (<u>Hodge Decomposition</u>). $(X, \langle,\rangle)$ compact Kähler.

Then $\exists$ orthogonal decompositions

$$A^{p,q}(X) = \partial A^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\partial}(X,\langle,\rangle) \oplus \partial^* A^{p+1,q}(X)$$

$$= \bar\partial A^{p,q-1}(X) \oplus \mathcal{H}^{p,q}_{\bar\partial}(X,\langle,\rangle) \oplus \bar\partial^* A^{p,q+1}(X)$$

where $\mathcal{H}^{p,q}_{\alpha}(X,\langle,\rangle) = \ker(\Delta_\alpha : A^{p,q}(X) \to A^{p,q}(X))$.

The spaces $\mathcal{H}^{p,q}$ are finite-dimensional, and

$$\mathcal{H}^{p,q}_{\partial} = \mathcal{H}^{p,q}_{\bar\partial}.$$

<u>Cor</u> The canonical projection

$$\mathcal{H}^{p,q}_{\bar\partial} \longrightarrow H^{p,q}(X) := H^q(X, \Omega^p_X)$$

is an isomorphism.

Pf. $\beta \in \mathcal{H}^{p,q}_{\bar\partial} \Rightarrow \bar\partial\beta = 0$ so get map.

Now $[\beta] = 0$ in $H^{p,q} \Rightarrow \beta = \bar\partial\gamma$

$\Rightarrow \langle\beta,\beta\rangle = \langle\bar\partial\gamma, \beta\rangle = \langle\gamma, \bar\partial^*\beta\rangle = 0.$

$\Rightarrow \beta = 0$. So injective. Surjection as usual. $\square$

Cor $(X, \langle , \rangle)$ compact Kähler. Then

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

This decomp. doesn't depend on metric!

Also, using $H^*(X, \mathbb{C}) = H^*(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$,

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

pf. Have

$$H^k(X, \mathbb{C}) \cong \mathcal{H}^k \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q} \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

Suppose $\alpha \in \mathcal{H}^{p,q}(X, \langle , \rangle)$,

$\alpha' \in \mathcal{H}^{p,q}(X, \langle , \rangle')$

represent same class in $H^{p,q}(X)$.

Then $\alpha' = \alpha + \bar{\partial}\gamma$, say, and

$d\bar{\partial}\gamma = d(\alpha' - \alpha) = 0$ since α', α are $\Delta_{\bar{\partial}}$-harm, hence Δ-harmonic. Also, $\bar{\partial}\gamma \perp \mathcal{H}^k(X, \langle , \rangle)_{\mathbb{C}}$.

So, Hodge decomp. for d yields $\bar{\partial}\gamma \in \operatorname{im} d(A^{k-1}_{\mathbb{C}})$.

So $[\alpha'] = [\alpha]$ in $H^k(X, \mathbb{C})$.

Conjugation follows since Δ is a real operator. $\square$.

Ex X cpt Riemann surface. We know that $H^1(X,\mathbb{C}) \cong \mathbb{C}^{2g}$ where g is the genus of X.

Hodge decomp tells us

$$H^1(X,\mathbb{C}) = H^0(X,\Omega_X) \oplus H^1(X,\mathcal{O}).$$

$$\underset{H^{1,0}}{\parallel} \qquad \underset{H^{0,1}}{\parallel}$$

Also $\overline{H^{1,0}} = H^{0,1}$! So, each of these is g-dim'l.

"Clearly" $H^2(X,\mathbb{C}) = H^1(X,\Omega^1)$ by dimensional considerations.

Serre duality tells us $(H^{1,1})^* \cong H^0$.

Ex X a compact complex torus, so $X \cong \mathbb{C}^g/\Lambda$, Λ a lattice. Have $X \underset{\text{diffeo}}{\cong} (S^1)^{2g}$, so we know dimensions in H^* by Künneth:

$$H^*(X,\mathbb{C}) \cong \bigotimes_{i=1}^{2g} H^*(S^1,\mathbb{C}).$$

Easy to see we have lots of harmonic forms in flat metric — any constant-coefficient form. These give enough dimensions!

④ $dz_1, \dots, dz_g$, $d\bar{z}_1, \dots, d\bar{z}_g$ span $(T_x^*X) \otimes \mathbb{C}$ at each x. Write $(V^*)^{1,0}$, $(V^*)^{0,1}$ for these, where $V = \mathbb{C}^g$. Then

$$H^{p,q}(X) \cong \Lambda^p (V^*)^{1,0} \otimes \Lambda^q (V^*)^{0,1} ! \; !$$

<u>Note</u> These even form a subalgebra of all forms under exterior (wedge) product!

<u>Ex</u> $X = \mathbb{P}^n$. You saw that

$$H^*(X, \mathbb{C}) \cong \mathbb{C}[u]/(u^{n+1}), \quad \deg u = 2.$$

[Did you?]

But $\mathbb{P}^n$ is Kähler, so have $\omega \in H^{1,1}(\mathbb{P}^n, \mathbb{C})$, with $\underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ times}} = d\mathrm{vol}$, nonzero in $H^{2n}(\mathbb{P}^n, \mathbb{C})$,

So $\underbrace{\omega \wedge \dots \wedge \omega}_{k \text{ times}} \neq 0$ in $H^{2k}(\mathbb{P}^n, \mathbb{C})$

So $H^{2k}(\mathbb{P}^n, \mathbb{C}) = H^{k,k}(\mathbb{P}^n)$.