Prop (1) $H^0(\mathbb{P}^N, \mathcal{O}(d)) \cong \mathbb{C}[x_0,\dots,x_N]_d$.

(2) $H^i(\mathbb{P}^N, \mathcal{O}(d)) = 0$, $i \neq 0, N$.

(3) $H^N(\mathbb{P}^N, \mathcal{O}(d)) \cong \mathbb{C}[x_0,\dots,x_N]_{-d-N-1}$.

(1) we already computed. (2) & (3) best computed algebraically (GAGA!) using Čech coh. See e.g. Hartshorne. I urge you to read this! $\square$.

Apply this to Cor 3. Get

$$(\dagger) \qquad H^q(\mathbb{P}^N, \Omega^{p-1}) \cong H^{q+1}(\mathbb{P}^N, \Omega^p)$$

provided $H^q(\mathbb{P}^N, \mathcal{O}(-p)) = 0$, $H^{q+1}(\mathbb{P}^N, \mathcal{O}(-p)) = 0$.

Now for $p \geq 1$, this is true whenever $q+1 < N$, by (1) & (2) of Prop. OTOH, by part (3) of Prop,

$H^N(\mathbb{P}^N, \mathcal{O}(-p)) = 0$ unless $p \geq N+1$; but then in that case $\Omega^p = 0$. So, $(\dagger)$ always holds.

By decreasing p and q each by 1 repeatedly, we get

$$H^q(\mathbb{P}^N, \Omega^p) \cong \begin{cases} 0 & p \neq q \\ \mathbb{C} & p = q. \end{cases}$$

Interpretation in terms of alg. cycles / ω_{FS} ...

Now let $X \subseteq \mathbb{P}^N$ be a nonsingular hypersurface defined by a single homogeneous polynomial f of degree d.

Note Every hypersurface is defined by a single equation!
[Well, irreducible hypersurface is easiest to see.]

Hodge Numbers of a Hypersurface

We begin with a fundamental exact sequence or two.

Let $\mathcal{O}_X$ denote the sheaf of regular functions on X, but thought of as a sheaf on $\mathbb{P}^N$ via

$$\mathcal{O}_X(V) := \mathcal{O}_X(X \cap V) \quad \text{for open sets } V \subseteq \mathbb{P}^N$$

(this is really the direct image sheaf).

Given a fn. on $\mathbb{P}^N$, can restrict to X: get a sheaf homom.

$$\mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X ,$$

surjective [why?]. What's kernel? Functions on $\mathbb{P}^N$ that vanish on X — locally, such a thing is $g \cdot f$ where f is the defining equation of X.

Fact This is $\cong \mathcal{O}_{\mathbb{P}^N}(-d)$, sheaf of sections of the line bundle.

Cor Have short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-d) \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0.$$

Cor Have short exact sequence

$$0 \to \Omega^p_{\mathbb{P}^N}(-d) \to \Omega^p_{\mathbb{P}^N} \to \Omega^p_{\mathbb{P}^N}|_X \to 0$$

for $p \geq 0$.

[Explain?] Tensor $0 \to \mathcal{O}(-d) \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0$ with $\Omega^p_{\mathbb{P}^N}$,

One more exact sequence:

$$0 \to T_X \to T_{\mathbb{P}^N}|_X \to N_{X/\mathbb{P}^N} \to 0$$

sheaf of sections of normal bundle.

Get, dually,

$$0 \to N^*_{X/\mathbb{P}^N} \to \Omega^1_{\mathbb{P}^N}|_X \to \Omega^1_X \to 0.$$

Fact $N^*_{X/\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^N}(-d)|_X$.

So, $0 \to \mathcal{O}_{\mathbb{P}^N}(-d)|_X \to \Omega^1_{\mathbb{P}^N}|_X \to \Omega^1_X \to 0$ exact.

Now, we'll prove

<u>Prop</u> If $p+q < N-1$, $r \geq 0$, then

$$H^q(X, \Omega^p_X \otimes \mathcal{O}_{\mathbb{P}^N}(-r)|_X) = \begin{cases} H^q(\mathbb{P}^N, \Omega^p_{\mathbb{P}^N}) & \text{if } r=0 \\ 0 & \text{otherwise.} \end{cases}$$

Pf. By induction on p.

<u>$p=0$</u> Tensor $0 \to \mathcal{O}(-d) \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0$ with $\mathcal{O}_{\mathbb{P}^N}(-r)$

To get $0 \to \mathcal{O}_{\mathbb{P}^N}(-d-r) \to \mathcal{O}_{\mathbb{P}^N}(-r) \to \mathcal{O}_X(-r) \to 0$.

Use long exact coh. sequence and coh. of $\mathcal{O}_{\mathbb{P}^N}(n)$ to get

$$\cancel{H^q(\mathcal{O}_{\mathbb{P}^N}(-d-r))} \to H^q(\mathcal{O}_{\mathbb{P}^N}(-r)) \to H^q(\mathcal{O}_X(-r)) \to \cancel{H^{q+1}(\mathcal{O}_{\mathbb{P}^N}(-d-r))}$$

for $q+1 < N$.

<u>$p>0$</u> Using $0 \to \mathcal{O}_{\mathbb{P}^N}(-d)|_X \to \Omega^1_{\mathbb{P}^N}|_X \to \Omega^1_X \to 0$

and applying Λ^p, get

$0 \to \Omega^{p-1}_X(-d) \to \Omega^p_{\mathbb{P}^N}|_X \to \Omega^p_X \to 0$ exact.

Then l.e.c.s. gives (after $\otimes \mathcal{O}(-r)$)

$$\cdots \to H^q(\Omega^{p-1}_X(-d-r)) \to H^q(\Omega^p_{\mathbb{P}^N}(-r)|_X) \to H^q(\Omega^p_X(-r)) \to$$

$$\to H^{q+1}(\Omega^{p-1}_X(-d-r)) \to \cdots$$

By induction get $H^q(\Omega^{p-1}_X(-d-r)) = 0 = H^{q+1}(\Omega^{p-1}_X(-d-r))$

for $\underbrace{(q+1)+(p-1)}_{p+q} < N-1$, so

$$H^q(\Omega^p_{\mathbb{P}^N}(-r)|_X) = H^q(\Omega^p_X(-r)) \text{ for } p+q < N-1.$$

To complete Prop, then, need

$$H^q(\Omega^p_{\mathbb{P}^N}(-r)|_X) = \begin{cases} H^q(\mathbb{P}^N, \Omega^p_{\mathbb{P}^N}), & r=0 \\ 0, & r>0. \end{cases}$$

Use

$$0 \to \Omega^p_{\mathbb{P}^N}(-d-r) \to \Omega^p_{\mathbb{P}^N}(-r) \to \Omega^p_{\mathbb{P}^N}(-r)|_X \to 0 \text{ exact.}$$

Long exact coh. seq. gives

$$H^q(\Omega^p_{\mathbb{P}^N}(-d-r)) \to H^q(\Omega^p_{\mathbb{P}^N}(-r)) \to H^q(\Omega^p_{\mathbb{P}^N}(-r)|_X) \to H^{q+1}(\Omega^p_{\mathbb{P}^N}(-d-r)) \to \cdots$$

<u>Enough to show</u> $H^q(\mathbb{P}^N, \Omega^p_{\mathbb{P}^N}(-k)) = 0 \ \forall \ p+q < N$ and $k>0$.

[Why?].

For that, use

$$0 \to \Omega^p_{\mathbb{P}^N}(-k) \to \mathcal{O}(-p-k) \otimes \mathbb{C}^{\binom{N+1}{p}} \to \Omega^{p-1}_{\mathbb{P}^N}(-k) \to 0,$$

as before... use induction on p.

$p=0$ $H^q(\mathbb{P}^N, \mathcal{O}(-k)) = 0 \quad \forall\, q<N$. ✓.

<u>$p>0$</u>

$$H^q(\Omega^p(-k)) \to H^q(\mathcal{O}(-p-k))^{\binom{N+1}{p}} \to H^q(\Omega^{p-1}(-k)) \to$$
$$\to H^{q+1}(\Omega^p(-k)) \to H^{q+1}(\mathcal{O}(-p-k))^{\binom{N+1}{p}} \to \cdots$$

For $q+1 < N$ (true when $p+q<N$ since $p \geq 1$!), get

$$H^q(\mathcal{O}(-p-k)) = 0 = H^{q+1}(\mathcal{O}(-p-k)), \text{ so}$$

$$H^q(\Omega^{p-1}(-k)) = H^{q+1}(\Omega^p(-k)).$$

So, reduces us to (eventually) $H^i(\mathcal{O}(-k)) = 0$ if $i<N$.

$\square$.