Math 428, Homework 1

Problem 1. Show that $\mathbb{C}^n$ ($n \geq 1$) does not have any compact complex submanifolds of positive dimension.

Problem 2. Show that there are no nonconstant holomorphic maps from $\mathbf{P}^1$ to a complex torus.

Problem 3. Prove that the map given set-theoretically by

$$(V \subset \mathbb{C}^n) \mapsto (\wedge^k V \subset \wedge^k \mathbb{C}^n)$$

defines a holomorphic embedding $\mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}^N$ where $N = \binom{n}{k}$.

Problem 4. Let M be a real manifold. Prove that a holomorphic atlas, up to equivalence, is uniquely determined by its associated sheaf of holomorphic functions.

Problem 5. Define a subset C of $\mathbf{P}^2$ by

$$C = \{(x : y : z) \mid y^2 z = x^3 + axz^2 + bz^3\}.$$

For which values of $a, b \in \mathbb{C}$ is C a complex submanifold of $\mathbf{P}^2$?

Problem 6. Let $q(x, y, z)$ be a homogeneous polynomial of degree 2. Define a subset D of $\mathbf{P}^2$ by

$$D = \{(x : y : z) \mid q(x, y, z) = 0\}.$$

Prove that D is a complex submanifold of $\mathbf{P}^2$ isomorphic to $\mathbf{P}^1$ if and only if q is irreducible. [Hint: first answer the question: when is a polynomial q of the form $q(x, y, z) = a_0 x^2 + a_1 y^2 + a_2 z^2$ irreducible?]

Problem 7. Given a finite-dimensional complex vector space V, let $\mathbf{P}(V)$ denote the set of lines in V. The set $\mathbf{P}(V)$ is naturally a compact complex manifold (isomorphic to $\mathbf{P}^n$ if $\dim(V) = n + 1$).

Fix a finite-dimensional complex vector space V. Define a subset H of $\mathbf{P}(V) \times \mathbf{P}(V^*)$ by

$$H = \{(\ell, \lambda) \mid \lambda(\ell) = 0\}.$$

The subset H is called the *incidence variety*. Prove that H is a compact complex submanifold of $\mathbf{P}(V) \times \mathbf{P}(V^*)$. What is its dimension?

Problem 8. Define a subset S of $\mathbf{P}^1 \times \mathbf{P}^2$, using homogeneous coordinates $(a : b)$ on $\mathbf{P}^1$ and $(x : y : z)$ on $\mathbf{P}^2$, by

$$S = \{((a : b), (x : y : z)) \mid a(yz^2 + x^3 + xz^2 + z^3) + b(yz^2 + x^3 + 2xz^2 + 2z^3) = 0\}.$$

At which points does S fail to be a complex submanifold of $\mathbf{P}^1 \times \mathbf{P}^2$? What is the geometric significance of those points in terms of the plane cubics $yz^2 + x^3 + xz^2 + z^3 = 0$ and $yz^2 + x^3 + 2xz^2 + 2z^3 = 0$?

Problem 9. Compute the Betti numbers of $\mathbf{P}^n$.