Transcendental Alg. Geom

This course provides something of an intro to complex geometry — with an emphasis on Hodge theory.

The main characters are complex manifolds: more specifically complex projective varieties or, more generally, compact Kähler manifolds.

[Insert general discussion of why.]

I'll assume you know a few basics of smooth (ie. C^∞) manifolds and of several complex variables (more than you'll need is covered in the first couple of chapters of Narasimhan's short book Several Complex Variables).

<u>Def</u> A <u>holomorphic atlas</u> $\{(U_i, \varphi_i)\}$ on a smooth manifold M consists of an open cover $\{U_i\}$ and diffeomorphisms

$$\varphi_i : U_i \xrightarrow{\sim} \varphi_i(U_i) \subseteq \mathbb{C}^n$$

(where $\varphi_i(U_i)$ is open in $\mathbb{C}^n$)

such that for all i, j, the composite

$$\mathbb{C}^n \supseteq \varphi_i(U_i \cap U_j) \xrightarrow{\varphi_i^{-1}} U_i \cap U_j \xrightarrow{\varphi_j} \varphi_j(U_i \cap U_j) \subseteq \mathbb{C}^n$$

is holomorphic. We usually abuse notation and write $\varphi_j \circ \varphi_i^{-1}$, etc rather than explicitly restricting the domain. The composite $\varphi_j \circ \varphi_i^{-1}$ is a <u>transition function</u>.

Two atlases $\{(U_i, \varphi_i)\}$, $\{(U_i', \varphi_i')\}$ are <u>equivalent</u> if $\varphi_j' \circ \varphi_i^{-1}$ is holomorphic for all i, j.

A <u>complex manifold</u> is a smooth manifold with a choice of equivalence class of holomorphic atlases.

__Note__ Usually convenient to assume our atlas is __maximal__ : contains every __chart__ (V, φ) compatible with it.

__Def__ __holomorphic function__ $f : X \to \mathbb{C}$ is a fn. s.t. $f \circ \varphi_i^{-1} : \mathbb{C}^n \supseteq \varphi_i(V_i) \to \mathbb{C}$ is holomorphic for every chart (V_i, φ_i).

Basic Way of Organizing Data

Holomorphic functions form a __sheaf__ $\boxed{\mathcal{O}_X}$.

Suppose X is a top. space. Let $\mathrm{Top}(X)$ denote its category of open sets: objects are open sets, maps are inclusions $U \subset V \subset X$.

A __presheaf of abelian groups__ on X is a functor $\mathcal{F} : \mathrm{Top}(X)^{op} \longrightarrow \underline{\mathrm{Ab\,Gps}}$.

[lecturer: spell it out concretely].

<u>Examples</u> • Continuous functions from open sets of X to $\mathbb{R}$ or $\mathbb{C}$.

• Smooth functions to $\mathbb{R}$ or $\mathbb{C}$. (if X is a smooth manifold).

• Holomorphic functions (if X is a complex manifold).

All these examples have two extra properties:

If $U \subset X$ is open and $\{U_i\}$ is an open cover of U,

(1) and $f, g \in \mathcal{F}(U)$ (e.g. holo fns $U \to \mathbb{C}$)

and $f|_{U_i} = g|_{U_i}$ $\forall i$, then $f = g$.

(2) If $f_i \in \mathcal{F}(U_i)$ $\forall i$ and

$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ $\forall j$, then there exists $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i \ \forall i$.

[explain meaning concretely!]

<u>Remark</u> If $U, V \subset \mathbb{C}^n$ are open subsets, you can tell whether a map $\varphi : U \to V$ is holomorphic by "checking" whether $f \circ \varphi : U \to V \to \mathbb{C}$ is holomorphic for all holomorphic functions $f : V \to \mathbb{C}$.

<u>Exercise</u> Show that, as a consequence, the choice of a holomorphic atlas is the same as specifying the sheaf $\mathcal{O}_X$.

<u>Prop</u> Let X be a compact, connected complex manifold. Then $\mathcal{O}_X(X) = \mathbb{C}$, i.e. any holomorphic function $X \xrightarrow{f} \mathbb{C}$ is constant.

Pf. By compactness, f achieves a maximum. By maximum principle, f is constant on any chart (U, φ) s.t. f achieves a max. at $x \in U$. Since X is connected ... $\square$