What is a complex vector space?

__A__ A real vector space V with an $\mathbb{R}$-linear endomorphism $J: V \to V$ such that $J^2 = -\mathrm{Id}$ (secretly, J is mult by $i = \sqrt{-1}$ on V).

What is a $\mathbb{C}$-linear functional on V?

__A__ An element $\lambda \in V^* \otimes_{\mathbb{R}} \mathbb{C}$ such that $\lambda(Jv) = i\lambda(v)\ \forall v \in V$.

What is a $\mathbb{C}$-antilinear functional on V?

__A__ $\lambda \in V^* \otimes_{\mathbb{R}} \mathbb{C}$ s.t. $\lambda(Jv) = -i\lambda(v)\ \forall v \in V$.

Now, J acts on $V^* \otimes_{\mathbb{R}} \mathbb{C}$ by $\lambda \mapsto \lambda \circ J$.

Since $V^*_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C}$ is now a <u>complex</u> vector space (using $\mathbb{C}$) with J-action and $J^2 = -\mathrm{Id}$, eigenvalues of J

are all $\pm i$ and J can be made upper triangular... writing $J = D + N$, D diagonal and N nilpotent, $DN = ND$, $-I = J^2 = D^2 + 2ND + N^2$.

Assuming N has nonzero entries only in

$$\begin{pmatrix} 0 & * & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & 0 & & \ddots & * \\ & & & & 0 \end{pmatrix},$$

we get $ND = 0 \implies N = 0$.

So J can be diagonalized.

<u>Def</u> $(V^*)^{1,0}$ is the i-eigenspace of $V^*_{\mathbb{C}}$, i.e, $\mathbb{C}$-linear functionals.

$(V^*)^{0,1}$ is the $(-i)$-eigenspace.

<u>Note</u> V^* does not naturally have complex conjugation, but $V^* \otimes \mathbb{C}$ does (via $\mathbb{C}$) and then $\overline{(V^*)^{1,0}} = (V^*)^{0,1}$.

<u>Main Example</u> X a complex manifold.

Choose local complex coords $z_1, \ldots, z_m$ near $x \in X$. Write $z_j = x_j + i y_j$.

Get a basis $\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right\}$ for $T_x X$.

Let $\boxed{J\left(\frac{\partial}{\partial x_i}\right) = +\frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}.}$

Then $dz_j\left(J\frac{\partial}{\partial x_j}\right) = (dx_j + i\,dy_j)\left(J\frac{\partial}{\partial x_j}\right) = +i$

$$= i\,dz_j\left(\frac{\partial}{\partial x_j}\right)$$

$$dz_j\left(J\left(\frac{\partial}{\partial y_j}\right)\right) = (dx_j + i\,dy_j)\left(-\frac{\partial}{\partial x_i}\right) = -1$$

$$= i\,dz_j\left(\frac{\partial}{\partial y_j}\right).$$

Good!

Claim This doesn't depend on coords.

Pf. Let φ be a coord. change with $\varphi(0)=0$ (for simplicity). The coordinate change on the $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$, is given by the Jacobian matrix of φ.

Let $\varphi = (\varphi_1, \ldots, \varphi_n)$, where

$\varphi_k = \varphi_k(z_1, \ldots, z_m)$ is holomorphic.

Write $\varphi_k = u_k + i v_k$. Then the CR equations say
$$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j},$$
$$\frac{\partial v_k}{\partial x_j} = -\frac{\partial u_k}{\partial y_j}.$$

So
$$J(\varphi) = \left(\Phi_{jk}\right) \quad \text{where}$$

$$\Phi_{jk} = \begin{pmatrix} \frac{\partial u_k}{\partial x_j} & \frac{\partial u_k}{\partial y_j} \\ \frac{\partial v_k}{\partial x_j} & \frac{\partial v_k}{\partial x_j} \end{pmatrix} = \begin{pmatrix} a_{jk} & b_{jk} \\ -b_{jk} & a_{jk} \end{pmatrix}.$$

Now, in this block form, J is written
$$\begin{pmatrix} \Psi & 0 & & \\ 0 & \Psi & & \\ & & \ddots & \\ & & & \Psi \end{pmatrix}, \quad \Psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, $\Phi_{jk}\Psi = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} b & -a \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \Psi\Phi_{jk}.$$

So $J(\Phi)\cdot J = J\cdot J(\Phi)$, and so J doesn't depend on the chart!

Now, we get a decomposition

$$(T_x^*X)\otimes\mathbb{C} = (T^*)^{1,0}\oplus(T^*)^{0,1} \quad \forall x\in X,$$

the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts.

This decomposes all complex-valued 1-forms, too! Locally,

$$dz_i \in (T^*)^{1,0}, \quad d\bar{z}_i \in (T^*)^{0,1}$$

It also decomposes k-forms, using

$$\Lambda^k T_x^* X\otimes\mathbb{C} = \Lambda^k\left((T^*)^{1,0}\oplus(T^*)^{0,1}\right)$$

$$= \bigoplus_{r+s=k} \Lambda^r (T^*)^{1,0} \otimes \Lambda^s (T^*)^{0,1}$$

Things here look like

$$f \cdot dz_{j_1} \wedge \dots \wedge dz_{j_r} \wedge d\bar{z}_{k_1} \wedge \dots \wedge d\bar{z}_{k_s}.$$

where f is any $\mathbb{C}$-valued function.

These are "forms of type (r,s)."

Given a form ω, write $\pi_{p,q}(\omega)$ for its (p,q)-component.

Let $\mathcal{A}^{p,q}_X$ denote the sheaf of C^∞ (p,q)-forms, $\mathcal{A}^{p,q}(X)$ the global ones.

<u>Def</u> Given $\omega \in \mathcal{A}^{p,q}_X(U)$, write

$$\partial\omega = \pi_{p+1,q}(d\omega), \quad \bar{\partial}\omega = \pi_{p,q+1}(d\omega).$$

Prop

(1) $d = \partial + \bar\partial$.

(2) $\partial^2 = 0 = \bar\partial^2$, $\quad \partial\bar\partial = -\bar\partial\partial$.

Pf. In local holo. coords $z_1, \dots, z_n$, we have
$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$
$$\frac{\partial}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

$dz_j = dx_j + i\,dy_j$, $\quad d\bar z_j = dx_j - i\,dy_j$.

If $\omega = \sum a_{IJ}\, dz^I \wedge d\bar z^J$, then

$$d\omega = \sum \frac{\partial a_{IJ}}{\partial x_K} dx_K \wedge dz^I \wedge d\bar z^J + \sum \frac{\partial a_{IJ}}{\partial y_K} dy_K \wedge dz^I \wedge d\bar z^J$$

OTOH,

$$\partial\omega = \sum \frac{\partial a_{IJ}}{\partial z_k} dz_k \wedge dz^I \wedge d\bar{z}^J$$

$$= \frac{1}{2} \sum \left(\frac{\partial a_{IJ}}{\partial x_k} - i \frac{\partial a_{IJ}}{\partial y_k} \right) (dx_k + i dy_k) \wedge dz^I \wedge d\bar{z}^J$$

$$= \frac{1}{2} \sum \frac{\partial a_{IJ}}{\partial x_k} dx_k \wedge dz^I \wedge d\bar{z}^J + \frac{1}{2} \sum \frac{\partial a_{IJ}}{\partial y_k} dy_k \wedge dz^I \wedge d\bar{z}^J$$

$$+ \frac{i}{2} \left(\frac{\partial a_{IJ}}{\partial x_k} dy_k - \frac{\partial a_{IJ}}{\partial y_k} dx_k \right) dz^I \wedge d\bar{z}^J ,$$

$$\bar{\partial}\omega = \sum \frac{\partial a_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz^I \wedge d\bar{z}^J$$

$$= \frac{1}{2} \sum \left(\frac{\partial a_{IJ}}{\partial x_k} + i \frac{\partial a_{IJ}}{\partial y_k} \right) (dx_k - i dy_k) \wedge dz^I \wedge d\bar{z}^J$$

$$= \frac{1}{2} \sum \left(\frac{\partial a_{IJ}}{\partial x_k} dx_k + \frac{\partial a_{IJ}}{\partial y_k} dy_k \right) \wedge dz^I \wedge d\bar{z}^J$$

$$- \frac{i}{2} \left(\frac{\partial a_{IJ}}{\partial x_k} dy_k - \frac{\partial a_{IJ}}{\partial y_k} dx_k \right) dz^I \wedge d\bar{z}^J ,$$

Adding, $d\omega = \partial\omega + \bar{\partial}\omega$. $\square$

<u>Note</u> As for vector spaces, $\overline{A_X^{p,q}} = A_X^{q,p}$ in $A_{X,\mathbb{C}}^{p+q}$.

Remark An *almost-complex manifold* is a manifold with a smooth complex structure operator $J : TX \to TX$ (so $J^2 = -I$). It is *integrable* if it comes from a structure of complex manifold on X as above.

Fact $d = \partial + \bar{\partial}$ iff the almost-complex structure is integrable.

Def The sheaf Ω^p_X of holomorphic p-forms on a cplx manifold X is $\ker\left(\bar{\partial} : A^{p,0}_X \to A^{p,1}_X\right)$.

We are going to compute sheaf cohomology $H^q(X, \Omega^q)$ using Hodge theory.

First step $\bar\partial$ Poincaré lemma The complex

$$A_X^{p,0} \xrightarrow{\bar\partial} A_X^{p,1} \xrightarrow{\bar\partial} \cdots \to A_X^{p,n} \to 0$$

is exact except at $A_X^{p,0}$.

Note Since $d^2 = 0$,

$$0 = (\partial + \bar\partial)^2 = \underbrace{\partial^2}_{(2,0)} + \underbrace{\partial\bar\partial + \bar\partial\partial}_{(1,1)} + \underbrace{\bar\partial^2}_{(0,2)}$$

$$\Rightarrow \quad \partial^2 = 0 = \bar\partial^2.$$

Prop Let $U \subset \mathbb{C}$ be an open neighborhood of a closed ε-disk, $\overline{B}_\varepsilon \subset U$.

Suppose $\alpha \doteq f d\bar z \in A_X^{0,1}(U)$. Then the function

$$g(z) = \frac{1}{2\pi i} \int_{B_\varepsilon} \frac{f(w)}{w - z} \, dw \wedge d\bar w$$

on B_ε satisfies $\bar\partial g = \alpha$.

$\boxed{\textbf{Rmk} \text{ Integrand blows up at } w = z!}$

Proof. Let $w = x+iy$. Then $dw \wedge d\bar{w} = -2i\, dx \wedge dy$.

We may assume $B_\varepsilon = B_\varepsilon(0)$ (i.e. centered at 0).

Choose a C^∞ fn. $\psi : B_\varepsilon \longrightarrow \mathbb{R}$ with compact support in B_ε s.t. $\psi|_V \equiv 1$ for some neighborhood $0 \in V \subset B_\varepsilon$.

Let $f_1 = \psi \cdot f$, $f_2 = (1-\psi) f$, so

$$f = f_1 + f_2.$$

Let $g_i(z) = \frac{1}{2\pi i} \int_{B_\varepsilon} \frac{f_i(w)}{w-z} dw \wedge d\bar{w}$.

Since $f_2|_V \equiv 0$, $g_2(z)$ is well-defined for any $z \in V$. What about $g_1(z)$?

$$g_1(z) = \frac{1}{2\pi i} \int_B \frac{f_1(w)}{w-z} dw \wedge d\bar{w}$$

$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(w)}{w-z} dw \wedge d\bar{w}$$ since $\mathrm{supp}(f_1) \subset B$ is compact.

We now make a change of variables, $re^{i\theta} = w - z$. Then

$$dw \wedge d\bar{w} = d(re^{i\theta}) \wedge d(re^{-i\theta})$$

$$= (e^{i\theta}dr + ire^{i\theta}d\theta) \wedge (e^{-i\theta}dr - ire^{-i\theta}d\theta)$$

$$= ir\,d\theta \wedge dr - ir\,dr \wedge d\theta = -2ir\,dr \wedge d\theta.$$

So

$$\boxed{g_1(z) = -\frac{1}{\pi}\int_{\mathbb{C}} f_1(z+re^{i\theta})e^{-i\theta}\,dr \wedge d\theta.}$$

Since f_1 is compactly supported, this is well-defined. So g is well-defined on V. Choosing $V = B_{\varepsilon-\delta}(0)$ with $\delta \to 0^+$, we see that $g = g_1 + g_2$ is well-defined on B_ε.

Let's use the same splitting to compute $\bar{\partial} g$.

First, by inspection g_2 is holomorphic in z so $\frac{\partial g_2}{\partial \bar{z}} = 0$.

Using $$g_1(z) = -\frac{1}{\pi}\int_{\mathbb{C}} f_1(z+re^{i\theta})e^{-i\theta}\,dr\wedge d\theta\,;$$

we get
$$\frac{\partial g_1}{\partial \bar z}(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f_1}{\partial \bar z}(z+re^{i\theta})e^{-i\theta}\,dr\wedge d\theta\,.$$

Now
$$\frac{\partial f_1}{\partial \bar z}(z+re^{i\theta}) = \frac{\partial f_1}{\partial w}(z+re^{i\theta})\cancel{\frac{\partial(z+re^{i\theta})}{\partial \bar z}} + \frac{\partial f_1}{\partial \bar w}(z+re^{i\theta})\frac{\partial(\bar z+re^{-i\theta})}{\partial \bar z}$$
$$= \frac{\partial f_1}{\partial \bar w}(z+re^{i\theta})\,.$$

So $$\frac{\partial g_1}{\partial \bar z}(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f_1}{\partial \bar w}(z+re^{i\theta})e^{-i\theta}\,dr\wedge d\theta$$
$$= \frac{1}{2\pi i}\int_B \frac{\partial f_1}{\partial \bar w}\frac{dw\wedge d\bar w}{w-z}$$ (working backwards as before).

$$= \frac{-1}{2\pi i} \lim_{\delta \to 0^+} \int_{B \setminus B_\delta(z)} d\left(\frac{f_1}{w-z}\, dw\right)$$

Stokes

$$= \frac{1}{2\pi i} \lim_{\delta \to 0^+} \int_{\partial B_\delta(z)} \frac{f_1}{w-z}\, dw \qquad (\text{since } f_1|_{\partial B} \equiv 0)$$

$= f_1(z)$ by usual 1-variable complex analysis.

[$\displaystyle\int_{\partial B_\delta(z)} \frac{f_1}{w-z}\, dw = \int_0^{2\pi} f_1(z + \delta e^{i\theta})\, d\theta$, so

taking $\lim_{\delta \to 0^+} \ldots$].

This proves the proposition! $\square$

Several Variables

$\bar{\partial}$-Poincaré lemma Let U be an open nbhd of the closure of a bounded polydisk $B_\varepsilon \subset \overline{B_\varepsilon} \subset U \subset \mathbb{C}^n$.

If $\alpha \in A^{p,q}(U)$, and $\bar\partial\alpha = 0$, $q \geq 1$, then $\exists\ \beta \in A^{p,q-1}(B_\varepsilon)$ with

$$\alpha = \bar\partial\beta \text{ on } B_\varepsilon.$$

Pf. First, reduce to $p=0$ as follows.

Write $\alpha = \sum_{I,J} \alpha_{IJ}\, dz^I \wedge d\bar z^J =: \sum_I dz_I \wedge \alpha_I$

$\left(\text{so } \alpha_I = \sum_J \alpha_{IJ}\, d\bar z^J \in A^{p,q}(U)\right)$.

Then $\bar\partial\alpha = 0$ iff $\bar\partial\alpha_I = 0 \ \forall I$;

and if $\alpha_I = \bar\partial\left(\sum_K g_{IK}\, d\bar z_K\right)$ then

$$\bar\partial\left(\sum_{I,K} dz_I \wedge g_{IK}\, d\bar z_K\right) = \sum_I dz_I \wedge \bar\partial\left(\sum_K g_{IK}\, d\bar z_K\right)$$

$$= \alpha.$$

Having thus reduced to $\alpha \in A^{0,q}(U)$, $\bar\partial\alpha = 0$, write $\alpha = \sum f_I\, d\bar z^I$.

let k be smallest s.t. $d\bar{z}_i$ does not appear for $i > k$. Then

$$\alpha = \alpha_1 \wedge d\bar{z}_k + \alpha_2 ,$$ with no $d\bar{z}_i$'s in α_2 for $i \geq k$. Now

$$0 = \bar{\partial}\alpha = \bar{\partial}\alpha_1 \wedge d\bar{z}_k + \bar{\partial}\alpha_2 .$$

It follows that, writing $\bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i} d\bar{z}_i$,

$\bar{\partial}_i \alpha_2 = 0$ for $i > k$, (because such terms can't be cancelled by $\bar{\partial}\alpha \wedge d\bar{z}_k$, which contains $d\bar{z}_k$ while $\bar{\partial}_i \alpha_2$ doesn't)

and then also $\bar{\partial}_i \alpha_1 = 0$ for $i > k$ (can't be cancelled by $\bar{\partial}\alpha_2$).

So each f_I is holomorphic in $z_{k+1}, \dots, z_n$.

Now let
$$g_I(z) = \frac{1}{2\pi i} \int_{B_{\varepsilon_k}} \frac{f_I(z_1, .., z_{k-1}, w, z_{k+1}, ..)}{w - z_k} dw \wedge d\bar{w}$$

By the 1-dim'l $\bar\partial$-Poincaré lemma,

$\dfrac{\partial g_I}{\partial \bar z_k} = f_I$ on B_{ε_k}. Also, g_I is holomorphic in $z_{k+1}, \dots, z_n$ and C^∞ in the remaining variables.

If we let

$$\gamma = \sum_{k \in I} (-1)^{n_I+1} g_I \, d\bar z_{I \setminus \{k\}}$$

where $I = \{i_1, \dots, \underset{\uparrow}{k}, \dots\}$, k at the n_Ith place

then:
- $\bar\partial_i \gamma = 0$ for $i > k$,
- $\bar\partial_k \gamma = -\alpha_1 \wedge d\bar z_k$

Then $\alpha + \bar\partial\gamma$ is $\bar\partial$-closed, but

$\alpha + \bar\partial\gamma = \alpha_2$ no longer contains $d\bar z_i$'s, $i > k$. Repeating the process, we eventually find

$$(\alpha + \bar\partial\gamma_k) + \dots + \bar\partial\gamma_1 = 0,$$ as desired $\square$

Hermitian structures Let (V, J) be a complex vector space. A real inner product $\langle\,,\rangle$ on V is <u>compatible</u> with the complex structure J if

$$\langle Jv, Jw\rangle = \langle v, w\rangle \quad \forall\, v, w \in V.$$

The <u>fundamental form</u> associated to $(V, J, \langle\,,\rangle)$ is then

$$\omega(v, w) = \langle Jv, w\rangle.$$

<u>Lemma</u> Suppose $\langle\,,\rangle$ is positive definite on V and compatible with J. Then, via $\Lambda^2_{\mathbb{R}} V^* \subset \Lambda^2 V^* \otimes_{\mathbb{R}} \mathbb{C}$ and $\Lambda^{1,1} V^* = (V^*)^{1,0} \otimes (V^*)^{0,1} \subset \Lambda^2 V^* \otimes_{\mathbb{R}} \mathbb{C}$, we have $\omega \in \Lambda^2_{\mathbb{R}} V^* \cap \Lambda^{1,1} V^*$.

Pf. $\overset{\omega(v,w):=}{\langle Jv, w\rangle} = \langle J^2 v, Jw\rangle = -\langle v, Jw\rangle$

$= -\langle Jw, v\rangle = -\omega(w,v)$. So $\omega \in \Lambda^2_{\mathbb{R}} V^*$.

To see in which component of $\Lambda^2 V^* \otimes \mathbb{C}$ ω lies:

$$\Lambda^2 V^* \otimes \mathbb{C} = \underset{(2,0)}{\Lambda^2 (V^*)^{1,0}} \oplus \underset{(1,1)}{(V^*)^{1,0} \otimes (V^*)^{0,1}} \oplus \underset{(0,2)}{\Lambda^2 (V^*)^{0,1}},$$

we can just compare $\omega(Jv, Jw)$ to $\omega(v,w)$: indeed, if $\omega \in \Lambda^{p,q}$ then

$$\omega(Jv, Jw) = i^p \bar{i}^q \omega(v,w) = i^{p-q} \omega(v,w).$$

But $\omega(Jv, Jw) = \langle J^2 v, Jw\rangle = -\omega(w,v)$
$= \omega(v,w)$ as above, so $p-q=0$! $\square$

<u>Remark</u> $\omega(-, J-) = \langle , \rangle$ so

(V, J, ω) also determines $\langle , \rangle$.

Similarly J is completely determined by $\omega(v,w) = \langle Jv, w\rangle$ given ω and $\langle , \rangle$.

Lemma Let $(V, \langle\, ,\, \rangle)$ be a real (pos. def) inner product space with compatible complex structure J. Then $(v,w) = \langle v,w\rangle - i\omega(v,w)$ defines a positive definite Hermitian form on V:

i.e., • $(v,v) \geq 0$ with equality iff $v=0$.

• $(v,w) = \overline{(w,v)}$

• $(\lambda v, w) = \lambda(v,w) \quad \forall \lambda \in \mathbb{C}$.

Pf. Check. i: $(v,v) = \langle v,v\rangle \cancel{- i\omega(v,v)}$. ✓

$$\overline{(w,v)} = \langle w,v\rangle + i\omega(w,v)$$
$$= \langle v,w\rangle - i\omega(v,w). \quad ✓$$

$$(\lambda v, w) \overset{\text{def}}{=} (av + bJv, w) \quad \text{where } \lambda = a+bi$$

$$= a(v,w) + b(Jv,w)$$
$$= a(v,w) + b\big(\langle Jv,w\rangle - i\omega(Jv,w)\big)$$
$$= a(v,w) + b\big(\omega(v,w) + i\langle v,w\rangle\big)$$
$$= a(v,w) + ib(v,w). \qquad \square$$

Note By definition,
$$\omega = -\operatorname{Im}((\ ,\ )).$$

One extends also $\langle\ ,\ \rangle$ to a Hermitian form on $V \otimes_{\mathbb{R}} \mathbb{C}$ by
$$\langle v \otimes \lambda, w \otimes \mu \rangle_{\mathbb{C}} \overset{\text{def}}{=} \lambda \overline{\mu} \langle v, w \rangle.$$

Lemma $(V^*)^{1,0} \perp (V^*)^{0,1}$ in this Hermitian product $\langle\ ,\ \rangle_{\mathbb{C}}$.

Pf. We have two projection operators
$$V^* \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\pi_{1,0}} (V^*)^{1,0}, \qquad V^* \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\pi_{0,1}} (V^*)^{0,1}$$
given by
$$\pi_{1,0}(\varphi) = \frac{1}{2}(\varphi - i\varphi \circ J) \quad \text{and}$$
$$\pi_{0,1}(\varphi) = \frac{1}{2}(\varphi + i\varphi \circ J). \quad \text{Then}$$

$$4 \cdot \langle \pi_{1,0}\varphi, \pi_{0,1}\psi \rangle_{\mathbb{C}} = \langle \varphi - i\varphi J, \psi + i\psi J \rangle$$

$$= \cancel{\langle \varphi, \psi \rangle - \langle \varphi J, \psi J \rangle}$$

since J is compatible with $\langle , \rangle$.

$$\cancel{- i \langle \varphi J, \psi \rangle - i \langle \varphi, \psi J \rangle}$$

since

$$\langle \varphi J, \psi \rangle = \langle \varphi J J, \psi J \rangle = \langle -\varphi, \psi J \rangle. \qquad \square$$

Lemma Under the isomorphism

$$\mathrm{Hom}_{\mathbb{C}}((V,J), \mathbb{C}) \cong (V^*)^{1,0},$$

one has

$$\tfrac{1}{2}(\,,\,) \longleftrightarrow \langle \,,\, \rangle_{\mathbb{C}} \big|_{(V^*)^{1,0}}.$$

Pf. The identification is given by

$$\mathrm{Hom}_{\mathbb{C}}((V,J),\mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = V^* \otimes \mathbb{C} \xrightarrow{\pi_{1,0}} (V^*)^{1,0}.$$

$$\varphi \longmapsto \tfrac{1}{2}(\varphi - i \varphi \circ J) =: j(\varphi)$$

Then

$$\langle \varphi - i\varphi\circ J, \psi - i \psi \circ J \rangle = \langle \varphi, \psi \rangle + \langle \varphi J, \psi J \rangle$$
$$- i \langle \varphi J, \psi \rangle + i \langle \varphi, J\psi \rangle$$
$$= 2 \langle \varphi, \psi \rangle - 2 i \,\omega(\varphi, \psi) = 2 (\varphi, \psi).$$

Divide both sides by four to get

$$\langle j(\varphi), j(\psi) \rangle_{\Phi} = \frac{1}{2} (\varphi, \psi). \qquad \square.$$