Examples

We next turn to examples of complex manifolds.

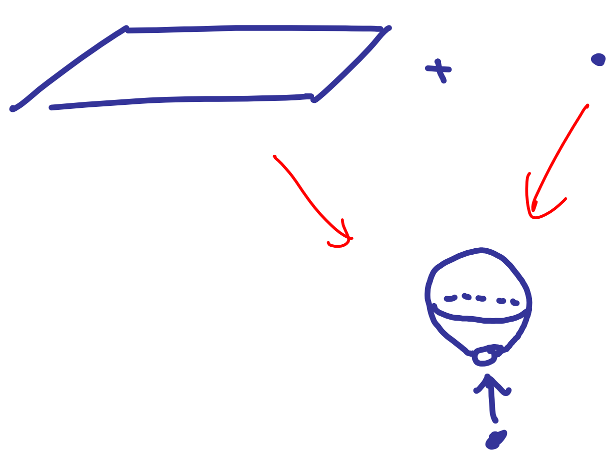
(0) Domains in $\mathbb{C}^n$.

[function theory dominates]

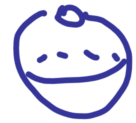
Rmk Recall that, unlike for C^∞ geometry, there are big differences between things that "look alike" — e.g. the unit disk in $\mathbb{C}$ is not biholomorphic to $\mathbb{C}$.

(0.5) Complex projective line.

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}. \simeq_{\text{diffeo}} S^2$$



Charts: $\mathbb{C}$ and $\mathbb{C}$, glued

 z  w

by identifying $z = w^{-1}$.

(1) Complex projective space.

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$

where the group $\mathbb{C}^*$ acts by rescaling.

$=$ set of lines in $\mathbb{C}^{n+1}$.

Describe a line by <u>homogeneous coords</u>:

$\ell = (a_0 : a_1 : \dots : a_n)$ if ℓ is spanned by the vector $(a_0, \dots, a_n)$ (so homogeneous coords not unique!).

How to make $\mathbb{P}^n$ a complex manifold? Get a chart by choosing list of basis vectors $v_0, v_1, \dots, v_n \in \mathbb{C}^{n+1}$. These determine an open set $U_v \subset \mathbb{P}^n$ consisting of "lines transverse to the span of $v_1, \dots, v_n$": i.e. letting $V = \text{span}\{v_1, \dots, v_n\}$,

$$U_v = \{\ell \mid \ell \cap V = \{0\}\}$$
$$= \{\ell \mid \ell \oplus V = \mathbb{C}^n\}.$$

Then every line in U_v is spanned by a unique vector

$$v_0 + c_1 v_1 + \dots + c_n v_n, \quad c_i \in \mathbb{C}.$$

Get a chart

$$U_v \to \mathbb{C}^n$$
$$\langle v_0 + c_1 v_1 + \dots + c_n v_n \rangle \longmapsto (c_1, \dots, c_n).$$

Particular examples: standard basis (but reordering to put e_i first). Then get an open set U_i consisting of points with homogeneous coords.

$$(c_0 : c_1 : \dots : c_{i-1} : 1 : c_{i+1} : \dots : c_n).$$

$U_i \xrightarrow{\varphi_i} \mathbb{C}^n$ via

$$(c_0 : c_1 : \dots : c_{i-1} : 1 : c_{i+1} : \dots : c_n) \longmapsto (c_0, \dots, c_{i-1}, c_{i+1}, \dots, c_n).$$

Q What do transition fns look like?

E.g. "$\mathbb{C}^n \xrightarrow{\varphi_0^{-1}} U_0 \cap U_1 \xrightarrow{\varphi_1} \mathbb{C}^n$"

$$(c_1, \dots, c_n) \longmapsto (1 : c_1 : \dots : c_n)$$
$$\parallel$$
$$\left(\frac{1}{c_1} : 1 : \frac{c_2}{c_1} : \dots : \frac{c_n}{c_1}\right)$$

$$\longmapsto \left(\frac{1}{c_1}, \frac{c_2}{c_1}, \dots, \frac{c_n}{c_1}\right).$$

Clearly holomorphic on $\varphi_0(U_0 \cap U_1) = \{(c_1, \dots, c_n) \mid c_1 \neq 0\}$.

$\mathbb{P}^n$ is an n-dimensional complex manifold, hence a real $2n$-manifold.
It's <u>not</u> diffeo. to S^{2n}.

Alternative description: $\mathbb{C}^{n+1}$ has std. Hermitian inner product,

$$\langle z, w\rangle = \sum z_i \bar{w}_i .$$

Every nonzero $z \in \mathbb{C}^{n+1}$ can be rescaled to have length 1, uniquely up to $z \rightsquigarrow \lambda z$ where $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

Now

$$\mathbb{S} = \{ z \in \mathbb{C}^{n+1} \mid \langle z, z\rangle = 1\} \simeq_{\text{diffeo}} S^{2n+1},$$

and above tells us

$$\mathbb{P}^n \simeq_{\text{diffeo}} \mathbb{S}/S^1 .$$

Have a fiber bundle

$$\begin{array}{ccc} S^1 & \to & S^{2n+1} \\ & & \downarrow \\ & & \mathbb{P}^n \end{array} .$$

Ex $n=1$. Get Hopf fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

representing generator of $\pi_3(S^2) \cong \mathbb{Z}$.

Two (closely related) alternative descriptions of $\mathbb{P}^n$:

the real Lie group $U(n+1)$ and complex Lie group $GL_{n+1}(\mathbb{C})$ act transitively on $\mathbb{C}^{n+1} \setminus \{0\}$. Since they act by linear autos of $\mathbb{C}^{n+1}$, the actions commute with the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$, hence descend to actions on $\mathbb{P}^n$. Since the actions are transitive, it suffices to identify the stabilizers of the point $(1:0:\cdots:0)$, which consist of "block upper-triangular matrices":

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \in U(n+1) \text{ or } GL_{n+1}.$$

Denote this subgroup of $U(n+1)$ by L and of GL_{n+1} by P. Note that if

$$A = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \text{ is unitary, i.e.}$$

$$\langle Az, Aw\rangle = \langle z, w\rangle, \text{ i.e. } A^* = A^{-1},$$

then actually

$$A = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right),$$

so $L \cong U(1) \times U(n)$. Thus

$\mathbb{P}^n \cong GL_{n+1}(\mathbb{C})/P$ and $\cong U(n+1)/U(1)\times U(n)$.

(2) Complex Lie groups.

If G is a complex manifold endowed with holomorphic maps

$$G \times G \xrightarrow{m} G \quad \text{(multiplication)}$$
$$G \xrightarrow{i} G \quad \text{(inverse)}$$

satisfying usual axioms for a group, it is a <u>complex Lie group</u>.

subexamples: $(\mathbb{C}^n, +)$, $((\mathbb{C}^*)^n, \cdot)$, $GL_n(\mathbb{C})$, <u>not</u> $U(n)$.

(3) <u>Quotients</u> If X is a complex manifold, G is a complex Lie group, and $G \times X \xrightarrow{a} X$ is a holomorphic action map, we can <u>hope</u> to form the quotient X/G. As with C^∞-manifolds, we impose conditions to ensure it works:

Def The action $a : G \times X \to X$ is **free** if $\forall x \in X$, $g \cdot x = x \Rightarrow g = 1$.

The action is **proper** if the map

$$G \times X \xrightarrow{(a, pr_2)} X \times X$$
$$(g, x) \mapsto (g \cdot x, x) \quad \text{is}$$

a proper map (i.e. inverse images of compact sets are compact sets).

Prop If $G \times X \xrightarrow{a} X$ is a proper, free action of a complex Lie group on a complex manifold, then X/G is naturally a complex manifold in such a way that the projection

$$X \xrightarrow{\pi} X/G \quad \text{is holomorphic.}$$

Ex $\mathbb{C}^*$ acts properly on $\mathbb{C}^{n+1} \setminus \{0\}$ (and freely, of course!).