Kähler manifolds

Complex manifold X with compatible Riemannian metric $\langle , \rangle$ is Kähler if fundamental 2-form ω is closed, i.e. $d\omega = 0$.

Ex Any metric on a Riemann surface is Kähler.

Ex Fubini-Study metric

$X = \mathbb{P}^n$. On $U_i = \{(z_0 : \dots : z_n) \mid z_i \neq 0\}$

$\cong \mathbb{C}^n$ (with $(w_0, \dots, \hat{w_i}, \dots, w_n)$ coords)

$$\omega_i = \frac{i}{2\pi}\partial\bar\partial \log\left(\sum_{l=0}^{n} |w_l|^2\right) \in A^{1,1}(U_i), \quad (w_i = 1)$$

Equivalently, choosing any local section $\mathbb{P}^n \xrightarrow{v} \mathbb{C}^{n+1} \setminus \{0\}$, we have

$$\omega = v^*\Big(\frac{i}{2\pi}\partial\bar\partial \log\left(|v|^2\right)$$

Claim This is fundamental form of a compatible metric.

Pf. First, need $\omega_i|_{U_i \cap U_j} = \omega_j|_{U_i \cap U_j}$.

Taking $f_i = \sum \left|\frac{z_\ell}{z_i}\right|^2$ vs.

$$f_j \quad \sum \left|\frac{z_\ell}{z_j}\right|^2 = \sum \left|\frac{z_j}{z_i}\right|^2 \left|\frac{z_\ell}{z_j}\right|^2 = H_{ij} f_i$$

where $H_{ij} = \left|\frac{z_j}{z_i}\right|^2$. Now

$$\omega_j = \frac{i}{2\pi} \partial\bar\partial \log(H_{ij} f_i) = \frac{i}{2\pi} \partial\bar\partial \log(f_i) + \frac{1}{2\pi} \partial\bar\partial \log(H_{ij})$$

in general, $\partial\bar\partial \log(|H|^2)$ for holo. H vanishes:

$$\partial\bar\partial \log(H\bar{H}) = \partial\left(\frac{1}{H\bar{H}} H \bar\partial \bar{H}\right)$$

$$= \partial\left(\frac{1}{\bar{H}} \bar\partial \bar{H}\right) = 0 \text{ since } \frac{1}{\bar{H}} \bar\partial \bar{H} \text{ is anti-holo.}$$

Then, need to see ω is a fund. form.

Well, it's clearly of type $(1,1)$. It's real, since

$$\overline{\omega} = \frac{-i}{2\pi} \overline{\partial\bar{\partial} \log(\sim)}$$

$$= -\frac{i}{2\pi} \bar{\partial}\partial \log(\sim) = \frac{i}{2\pi} \partial\bar{\partial} \log(\sim)$$

[note the fn. $\log(\sim)$ is real-valued].

Finally,

Fact Suppose metric $\langle\,,\rangle_{\mathbb{C}}$ in basis $dz_i, d\bar{z}_i$ is $\frac{1}{2}\sum h_{ij}\, dz_i d\bar{z}_j$. Then

$$\omega = \frac{i}{2}\sum h_{ij}\, dz_i \wedge d\bar{z}_j.$$

So, enough to show matrix of ω, (h_{ij}) is pos. def.

Compute e.g.

$$\partial\bar\partial \log\left(1+\sum_{i=1}^{n}|w_i|^2\right) = \frac{1}{(1+\sum|w_i|^2)}\sum h_{i\bar j}\,dw_i\wedge d\bar w_j$$

with $h_{i\bar j} = (1+\sum|w_i|^2)\,\delta_{ij} - w_i\bar w_j$, so

$$u^t(h_{i\bar j})\bar u = \|u\|^2 + \|u\|^2\|w\|^2 - (u,w)(w,u)$$

$$= \|u\|^2 + \|u\|^2\|w\|^2 - |(u,w)|^2 > 0 \text{ when } u\neq 0$$

by Cauchy-Schwarz. □

Rmk FS metric is $U(n+1)$-invariant on $\mathbb{P}^n$. Invariant metrics under large symmetry groups tend to have nice properties.

Cor of Example Every quasiprojective cplx manifold is Kähler.

[Restriction of Kähler metric is Kähler: fund form pulls back to fund form]

Crucial local property

Prop $(X, J, \langle , \rangle)$ compatible. Then metric is Kähler iff $\forall x \in X$ $\exists$ local holo. coords $z_1, \ldots, z_n$ at x, $x : z_i = 0 \; \forall i$, s.t.

$$\langle , \rangle_{\mathbb{C}} = \sum dz_i \cdot d\bar{z}_i + O(|z|^2),$$

i.e. $\langle , \rangle_{\mathbb{C}} = \sum dz_i \cdot d\bar{z}_i$ at $z = 0$ and

$$\frac{\partial h_{ij}}{\partial z_k}(0) = 0, \quad \frac{\partial h_{ij}}{\partial \bar{z}_k}(0) = 0 \quad \forall i, j, k.$$

We say metric osculates Euclidean metric to second-order.

Idea Linear coord change always makes $\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i$ + linear in $z, \bar{z}$ + h.o.t.

$d\omega = 0$ allows to remove linear terms by a quadratic change of coords.

Specifically, if

$$\omega = \frac{i}{2}\sum_{i,j,k}\left(\delta_{ij} + a_{ijk} z_k + a_{ij\bar{k}}\bar{z}_{\bar{k}} + \text{h.o.t.}\right) dz_i \wedge d\bar{z}_j,$$

take

$$z_k = w_k + \frac{1}{2}\sum b_{k\ell m} w_\ell w_m$$

where

$$\begin{cases} b_{k\ell m} = b_{km\ell}, \\ b_{jki} = -a_{ijk}. \end{cases}$$

$\square$.

Kähler manifolds have two kinds of additional structure on forms, hence on de Rham cohomology. First kind: pure linear algebra, doesn't use $d\omega = 0$. Second kind: involves derivatives, <u>does</u> use $d\omega = 0$.

Linear algebraic structure on $(V, J, \langle , \rangle)$.

Fund. 2-form ω gives an operator,

Lefschetz operator,

$$L : \Lambda^{\bullet} V^*_{\mathbb{C}} \longrightarrow \Lambda^{\bullet} V^*_{\mathbb{C}}$$

$$\alpha \longmapsto \omega \wedge \alpha \ (= \alpha \wedge \omega).$$

- This is a real operator:

$$L(\Lambda^{\bullet} V^*) \subset \Lambda^{\bullet} V^*$$

(since ω is real).

- $L(\Lambda^{p,q} V^*) \subset \Lambda^{p+1,q+1} V^*$.

(since ω is of type $(1,1)$).

Have an adjoint operator:

$$\langle \Lambda \alpha, \beta \rangle = \langle \alpha, L\beta \rangle \quad \forall \alpha, \beta \in \Lambda^{\bullet} V^*.$$

Using Hodge star, get

$\Lambda = *^{-1} \circ L \circ *$; so

$$\Lambda(\alpha) = (-1)^{(n+1)(n-k+2)} * L * \alpha, \qquad \alpha \in \Lambda^k V^*_{\mathbb{C}}$$

Define also operator

$$H : \Lambda^\bullet V^*_{\mathbb{C}} \longrightarrow \Lambda^\bullet V^*_{\mathbb{C}} \quad \text{by}$$

$$H(\alpha) = (k-m)\alpha, \qquad \alpha \in \Lambda^k V^*_{\mathbb{C}},$$

$$\tfrac{1}{2} \dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V = m.$$

<u>Prop</u>

$$[H,L] = 2L, \quad [H,\Lambda] = -2\Lambda, \quad [L,\Lambda] = H.$$

Pf. $(HL-LH)(\alpha) = (k+2-m)L\alpha - L(k-m)\alpha = 2L\alpha$.

Similarly for Λ but using $*$ also.

For last one, note that

$$(V, J, \langle , \rangle) \cong (\mathbb{C}^n, \text{std inner prod})$$

via Gram-Schmidt: so, in coords.

$x_1,\dots,x_n, y_1,\dots,y_n$, ← these guys are orthonormal.

Letting $z_j = x_j + iy_j$, $w_j = u_j + iv_j$,

$$\mathrm{Re}\left(\sum z_j\bar{w}_j\right) = \sum x_j u_j + \sum y_j v_j = \langle (x_j,y_j),(u_j,v_j)$$

So $\omega = -\mathrm{Im}\left(\sum z_j\bar{w}_j\right) = -\sum(y_j u_j - x_j v_j)$

$$= \left(\sum dx_j\wedge dy_j\right)(z,w), \text{ i.e. } \boxed{\omega = \sum dx_j\wedge dy_j}$$

Note $dz_j\wedge d\bar{z}_j = (dx_j + i\,dy_j)(dx_j - i\,dy_j)$

$$= -2i\,dx_j\wedge dy_j \quad \text{so}$$

$$\boxed{\omega = \frac{i}{2}\sum dz_j\wedge d\bar{z}_j.}$$

Let $\omega_j = \frac{i}{2}\,dz_j\wedge d\bar{z}_j$, so $\omega = \sum_j \omega_j$,

$L_j = \omega_j\wedge$ —. Get $L = \sum L_j$,

$\Lambda = \sum \Lambda_j$, $\Lambda_j = *^{-1} L_j *$.

__Fact__ Let $V_j = \text{span}\{x_j, y_j\}$.

Then

$$L = \sum_j 1 \otimes \cdots \otimes 1 \otimes L_j \otimes \cdots \otimes 1, \text{ on } \Lambda^\bullet V_1 \otimes \cdots \otimes \Lambda^\bullet V_n,$$

$$\Lambda = \sum_j 1 \otimes \cdots \otimes \Lambda_j \otimes 1 \otimes \cdots \otimes 1.$$

Pf.

First one is clear since

$\omega_j \wedge \alpha \wedge \beta = \alpha \wedge \omega_j \wedge \beta$ for any 2-form ω_j.

For second, observe

$$\langle L_j \alpha, \beta \rangle = \langle \omega_j \wedge (\alpha_1 \otimes \cdots \otimes \alpha_n), \beta_1 \otimes \cdots \otimes \beta_n \rangle$$

$$= \langle \alpha_1 \otimes \cdots \otimes (\omega_j \wedge \alpha_j) \otimes \cdots \otimes \alpha_n, \beta_1 \otimes \cdots \otimes \beta_n \rangle$$

$$= \left(\prod_{i \neq j} \langle \alpha_i, \beta_i \rangle \right) \langle \omega_j \wedge \alpha_j, \beta_j \rangle$$

$$= \left(\prod_{i \neq j} \langle \alpha_i , \beta_i \rangle \right) \langle \alpha_j , \Lambda_j \beta_j \rangle$$

$$= \langle \alpha_1 \otimes \dots \otimes a_n , (1 \otimes \dots \otimes 1 \otimes \Lambda_j \otimes \dots \otimes 1) (\beta_1 \otimes \dots \otimes \beta_n) \rangle .$$

$$\square$$

So, clear that

$$[L_i , \Lambda_k] = \delta_{ik} \, 1 \otimes \dots \otimes 1 \otimes [L_i , \Lambda_i] \otimes \dots \otimes 1 .$$

What is $[L_i , \Lambda_i]$? On $\Lambda^0 V_i^*$ it's

$$(L_i \Lambda_i - \Lambda_i L_i)(1) = 0 - \Lambda_i(\omega_i) = -1 ,$$

on $\Lambda^2 V_i^*$ it's

$$(L_i \Lambda_i - \Lambda_i L_i)(\omega_i) = L_i(1) = \omega_i ,$$

on $\Lambda^1 V_i^*$ it's zero. So clearly $= H$ on $\Lambda^\bullet V_i$. Then

$$[L , \Lambda](\alpha_1 \otimes \dots \otimes \alpha_n) = \sum_i \alpha_1 \otimes \dots \otimes H(\alpha_i) \otimes \dots \otimes \alpha_n$$

$$= \sum_i (\deg \alpha_i - 2) \, \alpha_1 \otimes \dots \otimes \alpha_n = H(\alpha_1 \otimes \dots \otimes \alpha_n) ,$$

$$\square$$

Consider

$\mathfrak{sl}_2\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2\mathbb{C} \right\}$. This is a Lie algebra under $[\,,\,]$. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$[h,e] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2e.$$

$[h,f] = -2f$ similarly,

$[e,f] = h$. So these are the defining relations of $\mathfrak{sl}_2\mathbb{C}$ in the basis $\{e,f,h\}$.

<u>Cor</u> $\mathfrak{sl}_2\mathbb{C}$ acts via $e \mapsto L$, $f \mapsto \Lambda$, $h \mapsto H$ on $\Lambda^{\bullet} V_{\mathbb{C}}$.

What do finite-dim'l reps of $\mathfrak{sl}_2\mathbb{C}$ look like? e.g. $\Lambda^\bullet V_{\mathbb{C}}$?

- $\Lambda^k V_{\mathbb{C}}$ is $(k-n)$-weight space for H. (eigenvectors w/ that eigenvalue).

- Every finite-dim'l rep is a direct sum of irreducibles.

Why? Let

$$\mathfrak{su}(2) = \{ X \in M_2\mathbb{C} \mid tr(X) = 0,\ X + X^{*} = 0 \}$$

conjugate transpose

So $X = \begin{pmatrix} a & b \\ \bar{b} & -a \end{pmatrix}$ with $a \in \mathbb{R}$.

Note that $\dim_{\mathbb{R}} \mathfrak{su}(2) = 3$, and $\mathfrak{su}(2)\cdot\mathbb{C} = \mathfrak{sl}_2\mathbb{C}$,
so $\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_2\mathbb{C}$.

Thus $SU(2)$ is a real form of $\mathfrak{sl}_2\mathbb{C}$.

Have exponential map $\exp: \mathfrak{sl}_2\mathbb{C} \to SL_2\mathbb{C}$,

where $SL_2\mathbb{C} = \{X \in M_2\mathbb{C} \mid \det(X) = 1\}$;

by $\exp(X) = e^X := \sum_{n \geq 0} \frac{X^n}{n!}$.

Suppose $X \in \mathfrak{sl}_2\mathbb{C}$ and $gXg^{-1} = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$.

Then $gX^2g^{-1} = (gXg^{-1})^2 = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix}$, and

$$gX^{2k}g^{-1} = \begin{pmatrix} a^{2k} & 0 \\ 0 & a^{2k} \end{pmatrix},$$

$$gX^{2k+1}g^{-1} = \begin{pmatrix} a^{2k} & 0 \\ 0 & a^{2k} \end{pmatrix}\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$$

$$= \begin{pmatrix} a^{2k+1} & a^{2k}b \\ 0 & -a^{2k+1} \end{pmatrix}.$$ In particular,

$g\exp(X)g^{-1} = \begin{pmatrix} e^a & B \\ 0 & e^{-a} \end{pmatrix}$ and so

$\exp(X) \in SL_2\mathbb{C}$. Similarly if

$X \in \mathfrak{su}(2)$ then

$$\exp(X) \in SU(2) = \left\{ X \in M_2\mathbb{C} \;\middle|\; \begin{matrix} UU^* = I, \\ \det(U) = 1 \end{matrix} \right\}.$$

<u>Facts</u>

(1) $SU(2)$ is a compact group.

(2) Any finite-dimensional representation of $SU(2)$ splits as a direct sum of irreducibles. [Explain].

(3) There are natural bijections,

$$\begin{tikzcd}
\{\text{fin. dim'l reps of } \mathfrak{sl}_2\mathbb{C}\} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \{\text{fin-dim'l reps of } SU(2)\} \\
\{\text{fin-dim'l reps of } SL_2\mathbb{C}\} \arrow[r, leftrightarrow] & \{\text{fin-dim'l reps of } SU(2)\} \arrow[u]
\end{tikzcd}$$

<u>Cor</u> Any fin-dim'l rep of $\mathfrak{sl}_2\mathbb{C}$ is <u>completely reducible</u>.

That is, it splits as a direct sum of irreducible reps.

- Suppose $Hv = \lambda v$. Then

$$H\Lambda v = \Lambda H v - 2\Lambda v = (\lambda-2)\Lambda v,$$

so mult. by Λ decreases the weight.

So, taking $v, \Lambda v, \Lambda^2 v, \ldots$ eventually get zero. So $\exists$ some vector $v \neq 0$ with $\Lambda v = 0$: "lowest-weight vector."

Can show then, using relations, that $v, Lv, L^2 v, \ldots$ span a sub-representation, which equals V when V is irreducible. So we get:

Prop V a finite-dim'l irrep of $\mathfrak{sl}_2\mathbb{C}$. Then $\exists\, v \in V$ with

$$V = \bigoplus_{1 \leq k \leq \dim V} \mathbb{C} \cdot L^k v ,$$

each $L^k v$ an eigenvector for H.

Our description shows that this is a splitting of V into different eigenspaces of H — so, $\mathbb{C} \cdot v$ is uniquely determined as the lowest weight subspace, or equivalently as

$$\ker(\Lambda : V \longrightarrow V).$$

Now, consider $\Lambda^k V^*_{\mathbb{C}}$ as above.

let $P^j = \{ \alpha \in \Lambda^j V^*_{\mathbb{C}} \mid \Lambda\alpha = 0 \}$.

Prop

(1) $\Lambda^k V^*_{\mathbb{C}} \cong \bigoplus_{i \geq 0} L^i(P^{k-2i})$. "Lefschetz decomposition"

This decomposition is orthogonal with respect to $\langle , \rangle_{\mathbb{C}}$.

(2) If $k > n$, then $P^k = 0$.

(3) $L^{n-k} : P^k \longrightarrow \Lambda^{2n-k} V^*_{\mathbb{C}}$ is injective for $k \leq n$.

(4) $L^{n-k} : \Lambda^k V^*_{\mathbb{C}} \longrightarrow \Lambda^{2n-k} V^*_{\mathbb{C}}$ is bijective for $k \leq n$.

(5) If $k \leq n$, $P^k = \{ \alpha \in \Lambda^k V^*_{\mathbb{C}} \mid L^{n-k+1} \alpha = 0 \}$.

$$\begin{tikzcd}
\Lambda^{n-2} V^* \arrow[rrrr, bend left, "L^2 \ \cong"] & \Lambda^{n-1} V^* \arrow[rr, bend right, "\cong"', "L"] & \Lambda^{n} V^* & \Lambda^{n+1} V^* & \Lambda^{n+2} V^*
\end{tikzcd}$$

etc.

Pf. (1) Decomp. follows from our lowest-weight discussion using complete reducibility.

To prove it can be made orthogonal, suppose v_1, v_2 are orthogonal lowest-weight vectors. Then $\langle v_1, L^l v_2\rangle = \langle \Lambda v_1, L^{l-1} v_2\rangle = 0$.

Also, if $k, l \geq 1$,

$$\langle L^k v_1, L^l v_2\rangle = \langle \Lambda L^k v_1, L^{l-1} v_2\rangle$$

$$= \cancel{\langle L^k \Lambda v_1, L^{l-1} v_2\rangle} + \langle [\Lambda, L^k] v_1, L^{l-1} v_2\rangle$$

We now use a formula that can be proven by induction:

$$\boxed{[L^k, \Lambda](\alpha) = k(a-n+k-1)L^{k-1}\alpha \text{ for } \alpha \in \Lambda^a V^*_{\mathbb{C}}.}$$

So $\langle L^k v_1, L^l v_2\rangle = \text{const} \cdot \langle L^{k-1} v_1, L^{l-1} v_2\rangle$.

Repeating, we eventually get 0. So, choosing an orthogonal basis of lowest-weight vectors makes the decomposition orthogonal.

(2) Suppose $\alpha \in P^a$, $a > n$. Suppose $L^l \alpha = 0$ but $L^{l-1}\alpha \neq 0$. Then

$$\cancel{-\Lambda L^l \alpha} + \cancel{L^l \Lambda \alpha} = l(a-n+l-1)L^{l-1}\alpha \neq 0$$

$\Longrightarrow$ $l = 0$ or $a-n+l-1 = 0$

$\Longrightarrow$ $l=0$ if $a>n$.

(3) Similar to (2).

(4) Follows from the above. (5) similar. □

Better $\boxed{\text{Orange boxed formula}}$ shows precisely how Λ acts on $\mathbb{C}\cdot\sum_{k\geq 0} L^k v$ when v is a lowest weight vector. So, the representation is completely determined by the weight.

An argument similar to ours for (2) above shows that the weight has to be an integer if the rep is going to be finite-dim'l.

So, we would know everything about

finite-dim'l reps of $\mathfrak{sl}_2\mathbb{C}$ once we knew irred. representations with all possible weights.

Construction $SL_2\mathbb{C}$ acts on $\mathbb{C}[x,y]$

by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(x,y) = f(ax+by, cx+dy)$.

Note $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot f(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(Ax+By, Cx+Dy)$

$= f(aAx+aBy+bCx+bDy, cAx+cBy+dCx+dDy)$.

while $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \end{pmatrix}$ good!

Ex (1) $e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

$e^{tH} \cdot f(x,y) = f(e^t x, e^{-t} y)$.

$$\frac{d}{dt}\Big|_{t=0} \cdot (e^{tH} \cdot x^u y^v) = \frac{d}{dt}\Big|_{t=0} (e^t)^{u-v} x^u y^v$$

$= (u-v)\, x^u y^v$. This tells us the weight.

(2) $e^{tF} = \exp\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

$$\frac{d}{dt}\Big|_{t=0} e^{tf} \cdot x^u y^v = \frac{d}{dt}\Big|_{t=0} (x+ty)^u y^v$$

$$= u\, x^{u-1} y^{v+1}.$$

(3) $e^{te} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$

$$\frac{d}{dt}\Big|_{t=0} e^{te} \cdot x^u y^v = \frac{d}{dt}\Big|_{t=0} x^u (y+tx)^v$$

$$= v\, x^{u+1} y^{v-1}.$$

So, let $V_k = \mathbb{C}[x,y]_k = \{ \text{homogeneous polys of deg } k \}.$

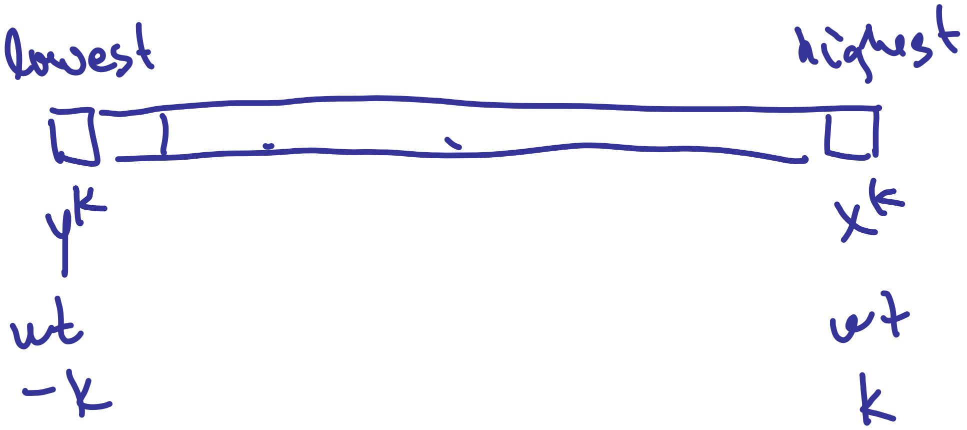
$\dim V_k = k+1$. Basis $v_j = x^j y^{k-j}$. Then

$$h \cdot v_j = (2j-k) v_j,$$

$$e \cdot v_j = (k-j) v_{j+1},$$

$$f \cdot v_j = j\, v_{j-1}.$$

So, v_0 is lowest weight vector, weight $-k$, $f \cdot v_0 = 0$. Moreover, $e^k \cdot v_0 = k!\, v_k$; $e \cdot v_k = 0$.



Thus, the reps $V_k = \mathbb{C}[x,y]_k \cong \mathrm{Sym}^k \mathbb{C}^2$ completely exhaust (up to isomorphism) the irreps of $\mathfrak{sl}_2\mathbb{C}$.

It's easy to see from the above the properties of the Lefschetz decomposition.

We've described how L and Λ behave on $\Lambda^\bullet V^*_{\mathbb{C}}$. How do they interact with the Hodge $*$? [Both $L^{\bullet}$ and $*$ provide reflections on $\Lambda^{\bullet}$]

Prop For $\alpha \in P^k \subset \Lambda^k V^*_{\mathbb{C}}$,

$$*L^j \alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} \mathbb{I}(\alpha)$$

where $\mathbb{I}(\alpha) = i^{p-q} \alpha$ when $\alpha \in \Lambda^{p,q}$.

Pf. See p 37-8 of Huybrechts. $\square$

Nice examples

$k=0$, $\alpha = 1$, $j=0$, get

$$d\mathrm{vol} = *1 = \frac{1}{n!} L^n 1 = \frac{\omega^n}{n!}.$$

$k=2p$, $\alpha \in P^{p,p}$, $j=0$:

$$*\alpha = \frac{(-1)^p}{(n-2p)!} L^{n-2p} \alpha = \frac{(-1)^p}{(n-2p)!} \omega^{n-2p} \wedge \alpha.$$

Rmk L, Λ, H are of bidegree $(1,1)$, $(-1,-1)$, $(0,0)$, so Lefschetz decomp. respects decomp by bidegree: if

$$P^{p,q} = P^{p+q}_{\mathbb{C}} \cap \Lambda^{p,q} V^*_{\mathbb{C}}, \text{ then}$$

$$P^k_{\mathbb{C}} = \bigoplus_{p+q=k} P^{p,q}.$$

This is useful: e.g.

$$\begin{array}{ccccc} \Lambda^2 V^*_{\mathbb{C}} = \Lambda^{2,0} V^* & \oplus & \Lambda^{1,1} V^*_{\mathbb{C}} & \oplus & \Lambda^{0,2} V^*_{\mathbb{C}} \\ \parallel & & \parallel & & \parallel \\ P^{2,0} & & P^{1,1} \oplus \mathbb{C}\cdot\omega & & P^{0,2} \end{array}.$$

Hodge Riemann bilinear relation

Hodge-Riemann pairing on $\Lambda^\bullet V^*_{\mathbb{C}}$, assoc to $(V, J, \langle , \rangle)$ is

$$Q(\alpha, \beta) = \langle (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k}, dvol \rangle$$

$\alpha, \beta \in \Lambda^k V^*_{\mathbb{C}}$. Note that

$Q(\alpha,\beta)\in\mathbb{C}$, but if $\alpha,\beta\in\Lambda^k_{\mathbb{R}}V^*$

then $Q(\alpha,\beta)\in\mathbb{R}$.

Prop (a) $Q(\Lambda^{p,q}V^*, \Lambda^{p',q'}V^*) = 0$ if $(p,q)\neq(p',q')$.

(b) $i^{p-q}Q(\alpha,\bar{\alpha}) = (n-(p+q))!\,\langle\alpha,\alpha\rangle_{\mathbb{C}}$ for $\alpha\in P^{p,q}$ with $p+q\leq n$.

Pf. (a) is clear since

$\mathrm{bideg}(\alpha\wedge\beta\wedge\omega^{n-k}) = \mathrm{bideg}\,\alpha + \mathrm{bideg}\,\beta + (n-k,n-k)$.

(b) $Q(\alpha,\bar{\alpha})\,dvol = (-1)^{\frac{k(k-1)}{2}}\,\alpha\wedge\bar{\alpha}\wedge\omega^{n-k}$

$= (-1)^{\frac{k(k-1)}{2}}\,\alpha\wedge L^{n-k}\bar{\alpha}$

$= (-1)^{\frac{k(k-1)}{2}}\,\langle\alpha, \overline{*^{-1}L^{n-k}\bar{\alpha}}\rangle_{\mathbb{C}}\,dvol$.

Now $\deg(L^{n-k}\bar{\alpha}) = 2n-2k+\underset{k}{\cancel{p+q,}}$

So $*^{-1}(L^{n-k}\bar{\alpha}) = (-1)^k * L^{n-k}\bar{\alpha}$

$$= (-1)^k (-1)^{\frac{k(k+1)}{2}} \frac{(n-k)!}{l!} i^{q-p} \bar{\alpha}$$

using

$$*L^j \alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} \mathbb{I}(\alpha) .$$

We get

$$Q(\alpha, \bar{\alpha}) = \cancel{(-1)^{k + \frac{k(k+1)}{2} + \frac{k(k-1)}{2}}} (n-k)!\, i^{q-p} \langle \alpha, \alpha \rangle_{\mathbb{C}}$$

or $\quad i^{p-q} Q(\alpha, \bar{\alpha}) = (n-k)! \langle \alpha, \alpha \rangle_{\mathbb{C}} . \quad \square$

Cor $i^{p-q} Q(\alpha, \bar{\alpha}) > 0$ for $\alpha \in P^{p,q}$, $\alpha \neq 0$.