Forms on Kähler Manifolds

<u>Recall</u> $(X, J, \langle , \rangle)$ cplx manifold w/ Riemannian metric. The metric is <u>Kähler</u> if $d\omega = 0$ where ω is the fundamental 2-form of the metric.

<u>We saw</u> Metric is Kähler iff it osculates the Euclidean metric to 2^{ND} order at each point.

<u>Alternative Prop</u> ω a real, closed (1,1)-form. Then locally $\exists$ a function u s.t.

$$\omega = \frac{i}{2} \partial \bar{\partial} u.$$

<u>Note</u> locally $\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} \, dz_i \wedge d\bar{z}_j$,

(h_{ij}) pos. def hermitian, so choice of factor in above is to make

u <u>real-valued</u>.

Pf. Restrict ω to a subset $U \cong$ to a ball in $\mathbb{R}^4$. Then $d\omega = 0 \Rightarrow$ $\exists\, \varphi \in A^1(U)$ (real-valued) s.t. $d\varphi = \omega$. Write $\varphi = \varphi_{1,0} + \varphi_{0,1}$, sum of pieces of type $(1,0)$ and $(0,1)$.

Note $\varphi = \bar{\varphi} \Rightarrow \overline{\varphi_{1,0}} = \varphi_{0,1}$.

Have

$$\omega = d\varphi = \underbrace{\partial\varphi_{1,0}}_{\in A^{2,0}} + \underbrace{\bar\partial\varphi_{1,0} + \partial\varphi_{0,1}}_{\text{type } (1,1)} + \underbrace{\bar\partial\varphi_{0,1}}_{\in A^{0,2}}$$

So $\bar\partial\varphi_{0,1} = 0$, $\partial\varphi_{1,0} = 0$.

Since $\varphi_{0,1}$ is $\bar\partial$-closed, if we make U a polydisk we get v s.t. $\bar\partial v = \varphi_{0,1}$.

Then $\partial\bar{v} = \overline{\bar\partial v} = \overline{\varphi_{0,1}} = \varphi_{1,0}$.

Now let $u = \frac{2}{i}(v - \bar{v})$. Then

$$\frac{i}{2}\partial\bar{\partial}u = \left(\frac{i}{2}\right)\frac{\partial\bar{\partial}v_1 - \bar{\partial}\partial(-\bar{v}_1)}{i/2} = \partial\varphi_{0,1} + \bar{\partial}\varphi_{1,0} = \omega,$$

as desired. [By construction, u is real-valued.] $\square$

u is called a <u>Kähler potential</u>, or a potential for the Kähler form ω.

Back to forms. From linear algebra, we get operators $L = \omega \wedge -$, $\Lambda = L^*$, H on differential forms.

We saw $\Lambda = *^{-1} \circ L \circ *$.

We're interested in the Dolbeault cohomology,

$$H^q(X, \Omega^p_X) = \frac{\ker(\bar{\partial} : A^{p,q}(X) \to A^{p,q+1}(X))}{\operatorname{Im}(\bar{\partial} : A^{p,q-1}(X) \to A^{p,q}(X))}.$$

We'll use same strategy as for H_{dR}.

Namely: let $\bar\partial^*$ be adjoint of $\bar\partial$, define $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$.

<u>Lemma</u> $d^* = \bar\partial^* + \partial^*$,

$(\partial^*)^2 = (\bar\partial^*)^2 = 0.$

<u>Main Proposition</u> (Kähler Identities).

$(X, J, \langle\,,\rangle)$ a Kähler manifold. Then:

(1) $[\bar\partial, L] = 0 = [\partial, L]$, $[\bar\partial^*, \Lambda] = 0 = [\partial^*, \Lambda]$.

(2) $[\bar\partial^*, L] = i\partial$, $[\partial^*, L] = -i\bar\partial$,

$[\Lambda, \bar\partial] = -i\partial^*$, $[\Lambda, \partial] = i\bar\partial^*$.

(3) $\Delta_{\bar\partial} = \frac{1}{2}\Delta = \Delta_\partial$,

(4) Δ commutes with $\partial, \bar\partial, \partial^*, \bar\partial^*, L, \Lambda$.