Corollary of Hodge Decomp

let $b_k(X)$ denote kth Betti number of X,

$$b_k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C}).$$

Then If X is compact Kähler,

$$b_{2k+1}(X) \text{ is even for all } k.$$

[Pf. Homework.]

<u>Cor</u> If X is compact Kähler and $\pi_1(X)$ is a free group then $\pi_1(X) \cong \{e\}$.

Pf. If F_k is a free group on k generators, then $F_k^{ab} \cong \mathbb{Z}^k$. Recall that

$$\pi_1(X)^{ab} \cong H^1(X, \mathbb{C}) \leftarrow \text{even dim'l.}$$

So $\pi_1(X)$ can't be free on an odd number of generators. But, it's known that a free gp on an <u>even</u> number of generators has a subgroup of finite index that's free on an odd number of generators. Such a subgroup of $\pi_1(X)$ corresponds to a finite covering space $\tilde{X}$ of X, which is also compact Kähler, and would have as its $\pi_1(\tilde{X})$ a free group on an odd number of generators... contradiction! $\square$

<u>Rmk</u> Even betti numbers of a compact Kähler manifold are nonzero. Indeed,

$$\frac{\omega^n}{n!} = d\mathrm{vol}, \text{ so } [\omega]^n \neq 0 \text{ in } H^{2n}(X,\mathbb{C}).$$

Hence $[\omega]^k$ is nonzero in $H^{2k}(X,\mathbb{C})$

$\forall k \leq \dim_{\mathbb{C}} X = n.$

<u>Picard group</u> Let X be compact Kähler. Then $\mathrm{Pic}(X)$ fits into an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^{1,1}(X)_{\mathbb{Z}} \to 0,$$

where $H^{1,1}(X)_{\mathbb{Z}} \overset{\text{def}}{=} i^{-1}(H^{1,1}(X))$ under $i: H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$.

and $\mathrm{Pic}^0(X)$ is a complex torus of dimension $h^{0,1}(X) = \dim_{\mathbb{C}} H^{0,1}(X)$.

Pf. • $\mathrm{Pic}^0(X)$ is a complex torus.

Exponential sequence gives

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) = \mathrm{Pic}(X)$$
$$\to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X).$$

This gives

$$0 \to \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} \to \mathrm{Pic}(X) \to H^2(X,\mathbb{Z}).$$

We need to know that $H^1(X,\mathbb{Z})$ is a lattice in $H^1(X, \mathcal{O}_X) = H^{0,1}(X)$.

Since $H^1(X, \mathbb{Z}) \subset H^{0,1}(X)$, $H^1(X, \mathbb{Z})$ is torsion-free, hence

$H^1(X, \mathbb{R}) \cong H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. We'll show that

$$\begin{tikzcd} H^1(X,\mathbb{R}) \arrow[r] \arrow[dr, "\sim"] & H^1(X,\mathbb{C}) \arrow[d] \\ & H^1(X,\mathcal{O}) \end{tikzcd}$$

is an isom

of real-vector spaces; that suffices.

But this follows from Hodge decomp: represent $[\alpha] \in H^1(X, \mathbb{R})$ by harmonic α, split $\alpha = \alpha_{1,0} + \alpha_{0,1}$, both harmonic, then the image of α in $H^{0,1}$ is $\alpha_{0,1}$; but $\alpha_{1,0} = \overline{\alpha_{0,1}}$ since α is real, so $\alpha_{0,1} = 0 \Rightarrow \alpha = 0$. Since $\dim_{\mathbb{R}} H^{0,1} = \dim_{\mathbb{C}} H^1(X, \mathbb{C}) = \dim_{\mathbb{R}} H^1(X, \mathbb{R})$, the map $H^1(X,\mathbb{R}) \to H^{0,1}(X)$ is an injection of vector spaces of same dim, hence an isomorphism.

- <u>Lefschetz Theorem on (1,1) classes</u>

$c_1(\mathrm{Pic}(X)) = H^{1,1}(X)_{\mathbb{Z}}$.

Pf. $\ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X))$

$= \ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}) \twoheadrightarrow H^2(X,\mathcal{O}_X))$

which certainly contains $H^{1,1}(X,\mathbb{Z})$,

so $H^{1,1}(X)_{\mathbb{Z}} \subseteq c_1(\mathrm{Pic}\, X)$.

Suppose

$\alpha \in c_1(\mathrm{Pic}(X)) = \ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X))$.

Write $[\alpha] \overset{\in H^2(X,\mathbb{C})}{=} [\alpha_{1,1}] + [\alpha_{2,0}]$ where

$\alpha_{1,1} \in \mathcal{H}^{1,1}$, $\alpha_{2,0} \in \mathcal{H}^{2,0}$.

Since $\overline{\alpha} = \alpha$, we get $\overline{\alpha_{2,0}} = \alpha_{0,2} = 0$,

so $\alpha = \alpha_{1,1}$, as desired. $\square$

<u>Def</u> The rank of the image
$$\mathrm{Pic}(X) \to \mathcal{H}^2(X, \mathbb{R})$$
[i.e. of $c_1(\mathrm{Pic}(X))$ modulo torsion]
is the <u>Picard number</u> of X.

<u>Rmk</u> In the above we implicitly used the following lemma:

<u>Lemma</u> The natural maps
$$H^k(X, \mathbb{C}) \to H^k(X, \mathcal{O}_X)$$
induced by sheaf homom. $\mathbb{C} \to \mathcal{O}_X$,
and $H^k(X, \mathbb{C}) \to \mathcal{H}^{0,k}(X)$ given by decomposition by bidegree, coincide.

Pf. The following diagram involving

resolutions commutes:

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, hook] & A_X^0 \otimes \mathbb{C} \arrow[r, "\partial"] \arrow[d, "\wr"] & A_X^1 \otimes \mathbb{C} \arrow[r, "\partial"] \arrow[d, "\pi_{0,1}"] & A_X^2 \otimes \mathbb{C} \arrow[r] \arrow[d, "\pi_{0,2}"] & \cdots \\
\mathcal{O}_X \arrow[r] & A_X^0 \otimes \mathbb{C} \arrow[r, "\bar\partial"] & A_X^{0,1} \arrow[r, "\bar\partial"] & A_X^{0,2} \arrow[r] & {}
\end{tikzcd}$$

$\square$.

<u>Formality</u> See Huybrechts, pp 145–6, for defs of <u>commutative differential graded algebras</u> and homomorphisms of them. A map $(A, d_A) \to (B, d_B)$ of cdgas is a <u>quasi-isomorphism</u> if the induced map $\mathcal{H}^\bullet(A) \to \mathcal{H}^\bullet(B)$ is an isomorphism.

A cdga (A, d_A) is <u>formal</u> if there exists a chain of homoms of cdgas

$$\begin{tikzcd}[column sep=small]
 & (B_1, d_{B_1}) \arrow[dl] \arrow[dr] & & (B_3, d_{B_3}) \arrow[dl] \arrow[dr] & & (B_k, d_{B_k}) \arrow[dl] \arrow[dr] & \\
(A, d_A) & & (B_2, d_{B_2}) & & \cdots & & (H^\bullet, 0)
\end{tikzcd}$$

which are quasi-isomorphisms and such that $(H^\bullet, 0)$ has zero differential.

Note that, applying $H^\bullet(-)$ to all these maps, get

$$H^\bullet(A) \cong H^\bullet.$$

Note also the directions of arrows! We do this because we'd like to think of quasi-isomorphisms as invertible, but

$$(A, d_A) \xleftarrow{\varphi} (B, d_B) \quad \text{a q.is}$$

(quasi-isomorphism)

needn't imply that there exists

$$(A, d_A) \longrightarrow (B, d_B) \quad \text{inducing}$$

the inverse map on cohomology.

__Note__ Formality is __not__ a trivial thing. You might naively try

$$\begin{tikzcd} & (\ker d_A, d_A) \arrow[dl] \arrow[dr] & \\ (A, d_A) & & (H^\bullet(A), 0) \end{tikzcd}$$

but the maps aren't quasi-isomorphisms!

__Thm__ [Deligne–Griffiths–Morgan–Sullivan] Let X be compact Kähler. Then

$$(A^\bullet(X)_{\mathbb{C}}, d) \quad \text{is formal.}$$

To prove it, let

$$d^c = -i(\partial - \bar\partial).$$

Then

$$\begin{aligned} dd^c &= -i(\partial + \bar\partial)(\partial - \bar\partial) \\ &= \cancel{-i\partial^2} - i(\bar\partial\partial - \partial\bar\partial) + \cancel{i\bar\partial^2} \\ &= 2i\partial\bar\partial = i(\partial\bar\partial - \bar\partial\partial) = i(\partial - \bar\partial)(\partial + \bar\partial) = -d^c d \end{aligned}$$

Lemma X compact Kähler, $\alpha \in A^k(X)_{\mathbb{C}}$.

(1) If $d\alpha = 0$ and $\alpha = d^c\gamma$, then there is $\beta \in A^{k-2}(X)_{\mathbb{C}}$ s.t. $\alpha = dd^c\beta$.

(2) If $d^c\alpha = 0$ and $\alpha = d\gamma$, then there is $\beta \in A^{k-2}(X)_{\mathbb{C}}$ s.t. $\alpha = d^c d\beta$.

Pf. (1) Write $\gamma = d\beta + \mathcal{H}(\gamma) + d^*\varphi$ via Hodge.

Since X is Kähler,

$$\partial\mathcal{H}(\gamma) = 0 = \bar{\partial}\mathcal{H}(\gamma).$$

So $$d^c\gamma = d^c d\beta + d^c d^*\varphi.$$

But also,

$$0 = d\alpha = \underbrace{dd^c d\beta}_{\cancel{-d^c d^2\beta}} + dd^c d^*\varphi = dd^c d^*\varphi.$$

Now, it follows from Kähler identities

$[\bar{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\bar{\partial}$ that

$[d^*, L] = -d^c$. Thus

$$d^c d^* = -(d^* L - L d^*) d^*$$
$$= -d^* L d^*,$$

$$-d^* d^c = +d^*(d^* L - L d^*)$$
$$= -d^* L d^*, \quad \text{i.e.}$$

$$d^c d^* = -d^* d^c.$$

Get
$$0 = d d^c d^* \varphi = -d d^* d^c \varphi$$

$$\Longrightarrow \quad 0 = \langle d d^* d^c \varphi, d^c \varphi \rangle = \langle d^* d^c \varphi, d^* d^c \varphi \rangle$$

$$\Longrightarrow \quad d^* d^c \varphi = 0 \Longrightarrow d^c d^* \varphi = 0.$$

So $\alpha = d^c \gamma = d^c d \beta$.

(2) Recall operator $\mathbb{I} = \sum_{p,q} i^{p-q} \pi_{p,q}$.

Check: α is d^c-closed, d-exact iff $\mathbb{I}(\alpha)$ is d-closed, d^c-exact. Use (1). $\square$.

Now, consider

$$(A^*(X)_{\mathbb{C}}, d) \xleftarrow{f_1} (\ker d^c, d) \xrightarrow{f_2} \left(\frac{\ker d^c}{\operatorname{im} d^c}, d\right).$$

Since $dd^c = -d^c d$, d preserves $\ker d^c$ and $\operatorname{im} d^c$, so maps are homoms of cdgas.

<u>Claim</u> f_1 is a quasi-isomorphism.

Pf. If $\alpha \in \ker d^c$ and $H^*(f_1)[\alpha] = 0$ in $H^*(X, \mathbb{C})$, so $\alpha = d\gamma$, then by lemma $\alpha = d^c d\beta = -dd^c\beta$, so $[\alpha] = d[-d^c\beta] = 0$ in $H^*(\ker d^c, d)$. So f_1 is injective on cohomology.

To see f_1 is surjective on cohomology, given $[\alpha] \in H^k(X, \mathbb{C})$, choose a harmonic representative α. Then $\bar\partial\alpha = 0 = \partial\alpha$, so $d^c\alpha = 0$, so $[\alpha] = H^\bullet(f_1)[\alpha]$. $\square$.

<u>Claim</u> f_2 is a quasi-isomorphism.

Pf. Suppose $\alpha \in \ker d^c \cap \ker d$ and $[\alpha] = 0$ in $H^\bullet(\ker d^c / \operatorname{im} d^c, d)$.

So $\alpha = d^c\varphi + d\psi$, $\psi \in \ker d^c$.

By lemma, $d\psi = d^c d\eta$ for some η, so $\alpha = d^c(\varphi + d\eta)$. Again by lemma, $\alpha = dd^c\tau$ for some τ, so $[\alpha] = 0$ in $H^\bullet(\ker d^c, d)$. Thus $H^\bullet(f_2)$ is injective.

Now, given $[\alpha] \in H^\bullet\left(\frac{\ker d^c}{\operatorname{im} d^c}, d\right)$, choose rep. $\alpha \in \ker d^c$. Then

$d\alpha$ is d-exact and d^c-closed,

so $d\alpha = dd^c\beta$ by Lemma; then

$\alpha - d^c\beta$ is d^c-closed and d-closed,

and $[\alpha - d^c\beta] = [\alpha]$ in $H^\bullet\left(\frac{\ker d^c}{\operatorname{im} d^c}, d\right)$.

But then $[\alpha] = H^\bullet(f_2)[\alpha - d^c\beta]$, so

$H^\bullet(f_2)$ is surjective. $\square$.

Finally,

<u>Claim</u> d acts trivially on $\frac{\ker d^c}{\operatorname{im} d^c}$.

Pf. $d^c\alpha = 0 \Rightarrow d\alpha$ is

d^c-closed and d-exact

$\Rightarrow d\alpha = d^c d\beta$ for some β

$\Rightarrow d\alpha = 0$ in $\frac{\ker d^c}{\operatorname{im} d^c}$. $\square$

Proves Theorem!

application Let

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{C}) \right\} \cong \mathbb{C}^3.$$

It's a Lie group (unipotent).
It has a subgroup

$$\Gamma(3) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{Z} + i\mathbb{Z} \right\}.$$

Let $I(3) = H_3 / \Gamma(3)$, the

3-dim'l. Iwasawa manifold.

Note

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+u & w+xv+z \\ 0 & 1 & y+v \\ 0 & 0 & 1 \end{pmatrix}.$$

So, $\left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ is a subgroup of $\Gamma(3)$

isomorphic to $\mathbb{Z} + i\mathbb{Z}$, and the quotient of $\Gamma(3)$ by this subgp is isomorphic to $(\mathbb{Z}+i\mathbb{Z})^2$. So

$I(3)$ is an $(S^1)^2$- bundle over $(S^1)^4$, topologically.

<u>Claim</u> $I(3)$ is <u>not</u> formal.

Indeed, we get cohomology classes α, β, γ from the invariant forms

$$dx, \; dy, \; dz - x\,dy \text{ on } H_3.$$

Suppose α, β, γ are cohomology classes with $0 = \alpha\beta$, $0 = \beta\gamma$, say

$$d\varphi = \alpha \wedge \beta, \qquad \beta \wedge \gamma = d\psi.$$

Then let

$$\langle \alpha, \beta, \gamma \rangle = \varphi \wedge \gamma - (-1)^{\deg \alpha} \alpha \wedge \psi.$$

Claim This is well-defined in cohomology.

It's the Massey triple product.

Fact If $(A^\bullet, d)$ is formal, all Massey triple products are zero.

Fact On $I(3)$,

$$\langle \alpha, \alpha, \beta \rangle = [-\alpha \wedge \gamma] \neq 0.$$

Note These notes are a bit incomplete. I'm happy to answer questions you may have...